XXX.

APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA*.

I PROPOSE to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the *Ausdehnungslehre*, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers*. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavours to determine the place of Quaternions and of what I have elsewhere† called Biquaternions in the more extended system, thereby *explaining* the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar* multi-

* [*American Journal of Mathematics Pure and Applied*, Vol. I. pp. 350—358.]

† *Proceedings of the London Mathematical Society* [*XX. supra*].

plication*; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *Scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

Consider now a system of n units $\iota_1, \iota_2, \ldots \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r, = \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the Ausdehnungslehre of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ; it is a thing totally different in kind from the vector ρ. The only way in

* [*American Journal of Mathematics*, Vol. I. p. 127 and p. 257, *supra*.]

which the factors q and ρ can be taken to be of the same kind, is to regard ρ as itself a special case of a quaternion, viz. a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations ρ and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors*; that is to say, they are operations which will turn a figure through a right angle in the three co-ordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points $\iota_0, \iota_1, \iota_2, \iota_3$, let us suppose that i turns the line $\iota_0\iota_2$ into $\iota_0\iota_3$; that j turns $\iota_0\iota_3$ into $\iota_0\iota_1$; and that k turns $\iota_0\iota_1$ into $\iota_0\iota_2$. The turning of $\iota_0\iota_2$ into $\iota_0\iota_3$ is equivalent to a translation along the line at infinity $\iota_2\iota_3$. We may, therefore, write $i = \iota_2\iota_3$, and so $j = \iota_3\iota_1$, $k = \iota_1\iota_2$. Now i turns $\iota_0\iota_2$ into $\iota_0\iota_3$; that is

$$i \,.\, \iota_0\iota_2 = \iota_0\iota_3,$$

or

$$\iota_0\iota_3 = \iota_2\iota_3 \,.\, \iota_0\iota_2 = -\iota_2^2 \,.\, \iota_0\iota_3.$$

We are therefore obliged to write $\iota_2^2 = -1$, and in a similar way we may find $\iota_1^2 = \iota_3^2 = -1$.

This at once enables us to find the rules of multiplication of the i, j, k. Namely, we have

$$jk = \iota_3\iota_1 \,.\, \iota_1\iota_2 = \iota_2\iota_3 = i,$$

$$ki = \iota_1\iota_2 \,.\, \iota_2\iota_3 = \iota_3\iota_1 = j,$$

$$ij = \iota_2\iota_3 \,.\, \iota_3\iota_1 = \iota_1\iota_2 = k,$$

and finally

$$ijk = \iota_2\iota_3 \,.\, \iota_3\iota_1 \,.\, \iota_1\iota_2 = -1.$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the square of each of our units equal to -1, as pointed out by Grassmann (*Math. Annalen*). But I venture to differ from his authority in thinking that the quaternion symbols do not in the first place answer to the "Elementargrösse" of the Ausdehnungslehre, but to

binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from factors of the same kind, and so of the product of two vectors, as a thing which might be thought of without regarding them as rectangular versors, grew upon Hamilton's mind, and led to the gradual replacement of the units i, j, k by the more general selective symbols S and V. To explain the laws of multiplication of i, j, k on this view, we must have recourse to the theory of "Ergänzung," or which comes to the same thing, *represent* an area ij by a vector k perpendicular to it. But the explanation in this case is by no means so easy; and it is instructive to observe that the distinction between a quantity and its "Ergänzung," i.e. between an area and its representative vector, which, for some purposes, it is so convenient to ignore, has to be reintroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors, which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

We have regarded the symbols i, j, k as rectangular versors operating on the quantities $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *rotors* (short for *rotators*) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation-velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A rotor is of the nature of the rotation-velocity of a rigid body, or of a force; it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b - a$); this is shewn by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols $\iota_1, \iota_2, \iota_3$ may be taken to mean unit vectors along the axes. In fact, if we write $\iota_0 + \iota_r = \alpha_r$, the points α will be situate on the axes at unit distance from the origin, and thus $\iota_r = \alpha_r - \iota_0$ will represent the unit vector from the origin to α_r.

The versors i, j, k will operate on these vectors in the same way as on the rotors $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. We find that

$$i\iota_2 = \iota_2\iota_3 \cdot \iota_2 = \iota_3, j\iota_3 = \iota_1, k\iota_1 = \iota_2.$$

These rules of multiplication coincide with those for i, j, k if we write the latter in place of ι_1, ι_2, ι_3. Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors ι_1, ι_2, ι_3 are identical with the areas $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$; it is only sometimes convenient to forget the difference between ι_1 and $\iota_2\iota_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points ι_0, ι_1, ι_2, ι_3 must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a *quadrant*. The product of four points $\alpha\beta\gamma\delta$ will then consist of three kinds of terms; (1) terms of the fourth order, being $\iota_0\iota_1\iota_2\iota_3$ multiplied by the determinant of the co-ordinates of the four points, which is proportional to $\sin(\alpha, \beta) \sin(\gamma, \delta) \cos(\alpha\beta, \gamma\delta)$; (2) terms of the second order, resulting from products of the form $\iota_0{}^2\iota_1\iota_2 = -\iota_1\iota_2$; (3) terms of order zero, resulting from products of the form $\iota_0^4\iota_0{}^2\iota_1{}^2$. Altogether we may arrange $\alpha\beta\gamma\delta$ in eight terms as follows:

$$\alpha\beta\gamma\delta = a + \Sigma b_{rs}\iota_r\iota_s + c\iota_0\iota_1\iota_2\iota_3. \quad (r, s \text{ different.})$$

And it is now easy to see that the product of any *even* number of linear factors will be of the same form. This form is what I have called a *biquaternion*, and may be easily exhibited as such. Namely, let us write ω for $\iota_0\iota_1\iota_2\iota_3$; then we have

$$i = \iota_2\iota_3,\ j = \iota_3\iota_1,\ k = \iota_1\iota_2,$$
$$\omega i = i\omega = \iota_1\iota_0,\ \omega j = j\omega = \iota_2\iota_0,\ \omega k = k\omega = \iota_3\iota_0,$$
$$\omega^2 = 1.$$

Therefore, the product of any even number of factors greater than two is a linear function of 1, $i, j, k, \omega, \omega i, \omega j, \omega k$; that is to say, it is of the form $q + \omega r$, where q, r are quaternions.

* Dr Klein's names for the Geometry of a space of uniform positive or negative curvature. [Cf. p. 191 *supra*.]

While the multiplication of ω with i, j, k is scalar, its multiplication with ι_0, ι_1, ι_2, ι_3 is polar. The effect of multiplying by ω is to change any system into its polar system in regard to the absolute.

The chief classification of geometric algebras is into those of *odd* and *even* dimensions. The geometry of an elliptic space of n dimensions is the same as the geometry of the points at an infinite distance in a flat or parabolic space of $n + 1$ dimensions; the theory of *points* and *rotors* in the former is the same as that of vectors and their products in the latter. Each requires a geometric algebra of $n + 1$ units. Thus the algebra of four units, leading as above to biquaternions, is either that of points and rotors in an elliptic space of three dimensions, or of vectors and their products in a flat space of four dimensions. All geometric algebras having an even number of units are closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points ι_0, ι_1, ι_2, ι_3 by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0$, $\omega\iota_1$, $\omega\iota_2$, $\omega\iota_3$ by the symbols 1, ωi, ωj, ωk, but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n *units.*

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$, such that $\iota_r{}^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain terms which are all of even order if m is even, and all of odd order if m is odd; for the substitution of -1 for any square factor of a term reduces the order of the term by 2.

A product of m units, all different, multiplied by any scalar is called a *term* of the order m. The sum of several terms of order m each multiplied by a scalar, is a *form* of order m. The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the m$^{\text{th}}$ *order is* $+1$ *or* -1 *according as the integer part of* $\frac{1}{2}(m+1)$ *is even or odd.* For the product $\iota_1\iota_2 \ldots \iota_m\iota_1\iota_2 \ldots \iota_m$ is transformed into $\iota_1^2\iota_2^2 \ldots \iota_m^2$ by $\frac{1}{2}m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2}m(m+1)$ is even or odd, which is equivalent to the rule stated.

The multiplication of a term P *of order* m *by a term* Q *of order* n, *having* k *factors common, is scalar or polar according as* mn $-$ k^2 *is even or odd.* Let $P = CP'$ and $Q = CQ'$, where C, P', Q' have no common factor; then the steps from $CP'CQ'$ to $CP'Q'C$, $CQ'P'C$, $CQ'CP'$ require respectively $k(n-k)$, $(m-k)(n-k)$, $k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn - k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k = 0$.)

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n = 2m + 1$, there are n terms of the order $n - 1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n - 2$ factors common.

We obtain in this way all the terms of the second order; and from them we can build up the terms of the fourth, sixth orders, &c. Let the product of all the units $\iota_1\iota_2 \ldots \iota_n$ be called ω, then these terms of the order $n-1$ shall be defined by the equations $k_r = \omega\iota_r$. It will follow that $k_1k_2 \ldots k_n = \mp 1$ according as m is even or odd, or, what is the same thing, according as the squares of the k are $+1$ or -1. By means of this formula, terms of order higher than m in the k, may be replaced by terms of order not higher than m. The multiplication of the k is always polar.

The terms of even order, regarded as compound units, constitute an algebra which is *linear* in the sense of Professor Peirce, viz. it is such that the product of any two of these terms is again a term of the system. The number of them is $2^{n-1} = 2^{2m}$; for the whole number of terms, odd and even, is

$$1 + n + \tfrac{1}{2}n(n-1) + \ldots + n + 1 = (1+1)^n = 2^n,$$

and the number of even terms is clearly equal to the number of odd terms.

I shall call the algebras whose units are the even terms formed with n elementary units $\iota_1\iota_2 \ldots \iota_n$, the n-*way geometric algebra*. Thus quaternions are the *three-way* algebra. We may regard the units of quaternions as expressed in either of two ways. First, in terms of the elementary units $\iota_1\iota_2\iota_3$; they are then $(1, \iota_2\iota_3, \iota_3\iota_1, \iota_1\iota_2)$. Secondly, we may write k_1, k_2 for the terms $\iota_2\iota_3$, $\iota_3\iota_1$, and the system may then be written $(1, k_1, k_2, k_1k_2)$. In this second form it is identical with the entire algebra of two elementary units, including both odd and even terms.

The five-way algebra depends upon the five terms k_1, k_2, k_3, k_4, k_5 and their products; the number of terms is sixteen. Now *we may obtain the whole of these sixteen terms by multiplying the quaternion set*

$$(1, k_1, k_2, k_1k_2)$$

by this other quaternion set

$$(1, k_4k_5, k_5k_3, k_3k_4).$$

For each of the sixteen products so obtained is a term of the even five-way algebra, and the products are all distinct. Moreover, *the two quaternion sets are commutative with one another.* For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients ωxyz are themselves quaternions of another set of units $(1, i_1, j_1, k_1)$, the i_1, j_1, k_1 being commutative with i, j, k.

I shall now extend this proposition, and shew that *the* $(2m+1)$ *way algebra is a compound of* m *quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_0 = k_1k_2$, and then

$$p_1 = k_1k_2k_6k_7 = p_0k_6k_7, \quad q_1 = k_3k_4k_5,$$
$$p_2 = p_1k_{10}k_{11} \quad , \quad q_2 = q_1k_8k_9,$$
$$\cdots\cdots\cdots\cdots \quad \cdots\cdots$$
$$p_r = p_{r-1}\,k_{4r+2}\,k_{4r+3} \quad , \quad q_r = q_{r-1}\,k_{4r}\,k_{4r+1}.$$

Consider now the quaternion sets

$$1,\ k_1,\ k_2,\ k_1k_2$$
$$1,\ k_4k_5,\ k_5k_3,\ k_3k_4$$
$$1,\ p_0k_6,\ p_0k_7,\ k_6k_7$$
$$1,\ q_1k_8,\ q_1k_9,\ k_8k_9$$
$$1,\ p_1k_{10},\ p_1k_{11},\ k_{10}k_{11}$$
$$\cdots\cdots\cdots\cdots$$
$$1,\ q_{r-1}k_{4r},\ q_{r-1}k_{4r+1},\ k_{4r}k_{4r+1}$$
$$1,\ p_{r-1}k_{4r+2},\ p_{r-1}k_{4r+3},\ k_{4r+2}k_{4r+3},$$
$$\cdots\cdots\cdots\cdots$$

viz.: a p-set and a q-set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical

with the terms of the $(2m+1)$ way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p-set and a q-set, because the q-set is of even order in the k, and no factors are common to the two sets. It remains only to examine the case of two p-sets and of two q-sets. Compare the two p-sets

$$1,\ p_{r-1}k_{4r+2},\ p_{r-1}k_{4r+3},\ k_{4r+2}k_{4r+3},$$
$$1,\ p_{s-1}k_{4s+2},\ p_{s-1}k_{4s+3},\ k_{4s+2}k_{4s+3},$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1}k_{4s+2}$, $p_{s-1}k_{4s+3}$, which are of odd order in the k; consequently the multiplication is scalar. The term $k_{4s+2}k_{4s+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q-sets

$$1,\ q_{r-1}k_{4r},\ q_{r-1}k_{4r+1},\ k_{4r}k_{4r+1},$$
$$1,\ q_{s-1}k_{4s},\ q_{s-1}k_{4s+1},\ k_{4s}k_{4s+1}.$$

Here again all the terms of the first sets are factors of $q_{s-1}k_{4s}$ and of $q_{s-1}k_{4s+1}$, and they have no factors in common with $k_{4s}k_{4s+1}$; since then all the terms are of even order in the k, the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: suppose this true for the first $m-1$ sets; that is to say, that no two of the products formed from them are identical or such that their product is $\pm k_1k_2 \ldots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1k_2 \ldots k_{2m-1}k_{2m}k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1}, so that it cannot be equal to ab. The product $abcd$ cannot be $\pm k_1 \ldots k_{2m+1}$ unless cd is $k_{2m}k_{2m+1}$ and ab is $k_1k_2 \ldots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first

$m-1$ sets are all distinct for the purposes of the $(2m-1)$ way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$ way algebra. But it is easy to see that the products of the first two sets are distinct.

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$ way algebra are also even terms of the $2m$-way algebra, and so also are their products by ω; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$-way algebra.

The multiplication of ω with the k of the $(2m-1)$ way algebra is scalar, because the k are factors in the ω, and they are both even terms.

Hence the $2m$-way algebra is a product of the $(2m-1)$ way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.

*XXXI.

BINARY FORMS OF ALTERNATE VARIABLES*.

Introduction.

1. Alternate numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; i.e., if $\bar{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \ldots$, $\bar{\beta} = b_1\beta_1 + b_2\beta_2 + \ldots$, where the a, b are scalars, and the α, β alternate numbers, then we shall have $\bar{\alpha}\bar{\beta} = -\bar{\beta}\bar{\alpha}$, and $\bar{\alpha}^2 = 0 = \bar{\beta}^2$. If M, N are homogeneous functions of alternate numbers of degrees m, n respectively, the number of interchanges of consecutive letters necessary to pass from MN to NM is mn; thus we have

$$MN = (-)^{mn} NM.$$

Or the product of two functions changes sign when the order of the functions is changed *only when their degrees are both odd;* that is to say, forms of odd degree among themselves behave like alternate numbers, forms of even degree in all cases like scalars. It follows that the square of any form of odd degree is zero.

Determinants of alternate numbers.

2. In expanding a determinant of alternate numbers, the order of the *rows* must be followed in multiplication; that is to say, in every term of the expanded determinant the

* [Communicated to the London Mathematical Society (June 12, 1879) by Dr Spottiswoode, P.R.S., and subsequently printed in the *Proceedings*, Vol. x. pp. 214—221. See note at end of this paper.]

constituent from the first row must be written first, that from the second row second, and so on. Thus, in expanding

$$\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3 \\ \nu_1, & \nu_2, & \nu_3 \end{vmatrix} = (\lambda\mu\nu),$$

the terms are of the form $\pm \lambda_r \mu_s \nu_t$, where rst is a permutation of 123 and the signs follow the ordinary rule. An interchange of two *columns* will then alter the sign of the determinant but an interchange of two rows will leave it unaltered. For the change of sign caused by the interchange is in the latter case counteracted by the change in the order of multiplication. Thus the determinant $(\lambda\mu\nu)$ written above is a symmetrical function of the $\lambda\mu\nu$.

3. Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space), and in that case it is convenient to regard the product of all the numbers of any set as equal to unity. Hence the products of all but one of the numbers make a new set, the *reciprocal* numbers. The n^{th} power of a determinant of *even* order n is $-\underline{|n}$; the $(n-1)^{\text{th}}$ power is the determinant of the reciprocal numbers. Considering especially determinants of the second order, we have an important theorem of their multiplication, viz., $(\lambda\mu)(\mu\nu) = -(\lambda\nu)$. For

$$\begin{aligned}(\lambda_1\mu_2 - \lambda_2\mu_1)(\mu_1\nu_2 - \mu_2\nu_1) &= \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &= (\lambda_2\nu_1 - \lambda_1\nu_2)\mu_1\mu_2 \\ &= -(\lambda_1\nu_2 - \lambda_2\nu_1).\end{aligned}$$

Hence $\quad (\lambda\mu)(\mu\nu)(\nu\rho)\dots(\sigma\tau)\ \{n \text{ factors}\} = (-)^{n-1}(\lambda\tau).$

An analogous theorem holds for determinants of the n^{th} order; viz., if we denote a determinant with r rows of λ and s rows of μ by $(\lambda^r\mu^s)$, where $r+s=n$; then

$$(\lambda^r\mu^s)(\mu^r\nu^s)\dots(\sigma^r\tau^s)\ \{k \text{ factors}\} = (-)^{s(n+r)k}(\lambda^r\tau^s) = (-)^{sk}(\lambda^r\tau^s),$$

since $n + r = 2r + s \equiv$ (mod. 2), and so $s(n+r) \equiv s^2 \equiv s$ (mod. 2).

Multipartite Forms.

4. A homogeneous function linear in each of n sets of k alternate numbers is called a k-ary multipartite form of the n^{th} order, or shortly, a k-ary form of the n^{th} order. We may consider also forms of any order lower than k in any of the sets; but for the present we restrict ourselves to the case in which the forms are linear. Consider now k forms, each linear in regard to the k alternate numbers $\lambda_1, \lambda_2, \dots \lambda_k$; viz.,

$$\begin{aligned}F_a &= a_1\lambda_1 + a_2\lambda_2 + \dots + a_k\lambda_k \\ &\vdots \\ F_k &= k_1\lambda_1 + k_2\lambda_2 + \dots + k_k\lambda_k,\end{aligned}$$

where the coefficients $a, b, \dots k$ are themselves k-ary multipartite forms of alternate numbers.

The product $F_aF_b \dots F_k = \Pi F$ is an invariant; that is to say, if for the λ we substitute k linear functions of them, say the μ, then the functions F will be transformed into functions of the μ; and if we form the same function of the new coefficients that ΠF is of the old coefficients, one will be equal to the other multiplied by the determinant of transformation.

For the product is $I . \lambda_1\lambda_2 \dots \lambda_k$, whether we regard the λ as linear functions of the μ or not; but in the latter case

$$\lambda_1\lambda_2 \dots \lambda_k = D . \mu_1\mu_2 \dots \mu_k,$$

where D is the determinant of transformation.

But it is also to a constant factor *près* the only function possessing this property. For let I be such a function, and calculate it for the linear forms $\lambda_1, \lambda_2, \dots \lambda_k$. As all the coefficients are here either unity or zero, I must be represented by a constant, say I_0. Now, expressing the λ in terms of the μ, we have, by hypothesis, $I_\mu = DI_0 = \lambda_1 \dots \lambda_k . I_0$.

It is to be remarked that the coefficients of transformation may themselves be forms involving alternate numbers to any *even* order. Otherwise the μ would not be alternate numbers, which is implied.

Moreover, we may regard the λ as no longer linear forms, but forms of any odd order; the new invariant will then be equal to the old one multiplied by a *commutant* of transformation. This leads to a useful theorem in the comparison of invariants; e.g.,

$$c \mid 456\vartheta \,.\, a \mid \vartheta 2 \,.\, b \mid \vartheta' 3 \,.\, c \mid 456\vartheta' = (c \mid 456\vartheta \mid)^2 \,.\, a \mid \vartheta' 2 \,.\, b \mid \vartheta' 3.$$

The proposition may be further extended by considering forms which involve the λ to an order higher than the first, but less than k; i.e., linear functions of their products r together. Let F_a, F_b, ... F_k be forms such that the sum of their orders in the λ is equal to k; then their product is an invariant (and the only one) in regard to linear transformations of the λ.

If the sum of the orders is less than k, $= h$ suppose, the product is a covariant; viz., it is a linear function of the products of the λ, h together, which, whether derived from the original forms before or after the λ are replaced by linear functions of the μ, has the same value. In this case the r-products of the λ are replaced by linear functions of the r-products of the μ, the coefficients being determinants of the r^{th} order formed with the coefficients of the μ.

Now suppose any number of forms $F_a \dots F_n$ to involve any number of sets of alternate numbers $\lambda\mu \dots \tau$, yet so that the sum of the orders in any one set is not greater than the number of alternates in that set; then the product of the forms is an invariant or covariant in regard to each of the sets taken separately, in the sense explained above; and it is the only function which possesses this property.

Expression of Unsymmetrical forms in terms of Symmetrical forms and Determinants of the Variables.

5. The binary form in two sets of variables,

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = a12,$$

will be called *symmetrical* when $a_{12} = a_{21}$. In that case the

interchange of the variables λ, μ only alters the sign of the form; we have $a12 = -a21$. We have, in general,

$$a12 + a21 = (a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1) = s(\lambda\mu).$$

The factor $a_{12} - a_{21}$ is an invariant when both sets of variables are transformed by the same substitutions; in fact, we have

$$(\lambda_1\mu_2 - \lambda_2\mu_1)\, a12 = a_{21} - a_{12} = -s,$$

which exhibits it as a product of the form by the universal covariant $(\lambda\mu)$ or (12). We may make a symmetrical form from $a12$ by adding $-a21$ to it; half this sum shall be called the mean value of $a12$ and denoted by $\overline{a12}$. Thus, we have

$$\overline{a12} = \tfrac{1}{2}(a12 - a21) = a_{11}\lambda_1\mu_1 + \tfrac{1}{2}(a_{12} + a_{21})(\lambda_1\mu_2 + \lambda_2\mu_1) + a_{22}\lambda_2\mu_2;$$

but also

$$\tfrac{1}{2}(a12 + a21) = \tfrac{1}{2}(a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1),$$

therefore
$$a12 = \overline{a12} + \tfrac{1}{2}s\,.\,(12),$$
where
$$-s = (12)\, a12.$$

It is easy to apply this to forms involving more sets of variables, if we remember that in these results the coefficients may themselves be such forms. We have, for example,

$$a123 = \overline{a123} - \tfrac{1}{3}\{(23)\,a\,.\,(23) + (13)\,a\,.\,(13) + (12)\,a\,.\,(12)\},$$

and so, generally,

$$a12\dots k = \overline{a12\dots k} - \frac{\Sigma\,(12)\,a\,.\,(12)}{n} + \frac{\Sigma\,(12)\,(34)\,a\,.\,(12)\,(34)}{\tfrac{1}{2}n\,(n-1)} - \dots,$$

the coefficients being the reciprocals of the binomial coefficients*.

Theory of Quadratic Forms.

6. In connection with the quadratic form $a12$, we have already considered the invariant

$$a12 + a21 = s_a\,(12), \text{ where } (12)\,a12 = -s_a.$$

We have thus the formula

$$a21 = -a12 + s_a\,.\,(12) \quad\dots\dots\dots\dots\dots\dots\dots (1).$$

* [In the last two equations of Par. 5, the symbols a are to be considered as abbreviations for $a23$, $a13$, $a12$; $a1234$, &c. Sp.]

To this if we add

$$(13)\,a12 = -a32, \quad (23)\,a12 = -a13 \ldots\ldots\ldots\ldots (2),$$

we shall have exhausted all the invariants and covariants of the first order in the coefficients.

The *discriminant* which is the only invariant of the second order, is given by the square of the quadratic form. We may write

$$a12\,.\,a12 = -2D_{aa} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where

$$D_{aa} = a_{11}a_{22} - a_{12}a_{21}.$$

We have

$$\begin{aligned} a21\,.\,a21 &= \{-a12 + s_a\,.\,(12)\}\,\{-a12 + s_a\,.\,(12)\} \\ &= a12\,.\,a12 - 2s_a\,(12)\,a12 + s_a^2\,.\,(12)^2 \\ &= a12\,.\,a12 + 2s_a^2 - 2s_a^2 = a12\,.\,a12, \end{aligned}$$

as it obviously should; and this may be regarded as a proof of the formula (2).

Moreover,

$$\begin{aligned} a12\,.\,a21 = a12\,\{-a12 + s_a\,(12)\} &= -a12\,.\,a12 - s_a^2 \\ &= 2D_{aa} - s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4); \end{aligned}$$

and again since

$$\begin{aligned} \overline{a12} &= a12 - \tfrac{1}{2}s\,.\,(12), \\ \overline{a12}\,.\,\overline{a12} &= a12\,.\,a12 + s_a^2 + \tfrac{1}{4}s_a^2\,(12) \\ &= a12\,.\,a12 + \tfrac{1}{2}s_a^2; \end{aligned}$$

therefore

$$\overline{D_{aa}} = D_{aa} - \tfrac{1}{4}s_a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $\overline{D_{aa}}$ is the corresponding invariant of the symmetrical function $\bar{a}$.

Since the product of two *even* forms is independent of their order, we have

$$a12\,.\,a13 = a13\,.\,a12.$$

But

$$\begin{aligned} a12\,.\,a13 + a13\,.\,a12 &= -(23)\,a12\,.\,a13\,.\,(23) \\ &= a13\,.\,13\,(23) = -2D_{aa}\,(23), \end{aligned}$$

therefore

$$a12\,.\,a13 = -D_{aa}\,(23) \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence

$$a12\,.\,a31 = a12\,\{-a13 + s_a\,(13)\} = D_{aa}\,.\,(23) - s_a\,.\,a32 \ldots\ldots (7),$$

$$\begin{aligned} a21\,.\,a31 &= \{-a12 + s_a\,(12)\}\,\{-a13 + s_a\,(13)\} \\ &= -D_{aa}\,(23) + s_a\,(a32 + a23) - s_a^2\,(23) \\ &= -D_{aa}\,(23) = a12\,.\,a13 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8), \end{aligned}$$

$$a21\,.\,a13 = a21\,\{-a31 + s_a\,(13)\} = D_{aa}\,(23) - s_a\,.\,a23 \ldots (9).$$

In the case of a symmetrical function $s = 0$, and the formulæ reduce themselves to the following:

$$a12\,a12 = -2D_{aa}, \quad a21 = -a12, \quad a12\,.\,a13 = -D_{aa}\,.\,(23).$$

The following formulæ are also worth noticing:

$$\begin{aligned} a12\,.\,a34 + a13\,.\,a24 &= \{D_{aa}\,.\,(14) - s_a\,.\,a14\}\,(23) \ldots\ldots (10), \\ a12\,.\,a34 - a21\,.\,a43 &= s_a\,\{a23\,.\,(14) - a14\,.\,(23)\} \\ &= s_a\,\{a41\,.\,(23) - a32\,.\,(14)\} \ldots\ldots (11); \end{aligned}$$

these enable us to make a single or bifid substitution in a product of the form by itself with two different sets of variables.

Passing now to two different forms a, b, we have in the first place the covariant

$$\vartheta_{ab}23 = a12.b13 = \left(\begin{vmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12} \end{vmatrix}\!\!\big(\mu_1\mu_2\big)\!\!\big(\nu_1\nu_2\big)\right) \ldots (12),$$

and next the invariant

$$D_{ab} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = -a12\,.\,b12 \ldots\ldots (13).$$

We may now make the following analysis of the one-place and two-place products of a and b:

$$\left.\begin{aligned} a12\,.\,b13 &= \vartheta 23, \\ a12\,.\,b31 &= a12\,\{-b13 + s_b\,(13)\} = -\vartheta 23 - s_b\,.\,a32, \\ a21\,.\,b13 &= \{-a12 + s_a\,(12)\}\,b13 = -\vartheta 23 - s_a\,.\,b23, \\ a21\,.\,b31 &= \{-a12 + s_a\,(12)\}\,\{-b13 + s_b\,(13)\} \\ &= \vartheta 23 + s_b\,.\,a32 + s_a\,.\,b23 - s_a s_b\,.\,23 \end{aligned}\right\} (14),$$

$$\left.\begin{aligned} a12\,.\,b12 &= -D_{ab}, \\ a12\,.\,b21 &= a12\,\{-b12 + s_b\,.\,(12)\} = D_{ab} - s_a s_b \end{aligned}\right\} \ldots\ldots (15).$$

Let us now apply to ϑ all those formulæ which we have proved for a single form. Thus from (1), we have

$$\vartheta 23+\vartheta 32=-(23)\vartheta 23(23)=-(23)a12.b13.(23)=a13.b13.(23)$$
$$=D_{ab}(23) \text{ or } s_\vartheta=D_{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Next, from (3),

$$-2D_{\vartheta\vartheta}=\vartheta 23 \,.\, \vartheta 23=a12 \,.\, b13 \,.\, a42 \,.\, b43=D_{aa}D_{bb}\,(14)^2$$
$$=-2D_{aa}D_{bb} \text{ or } D_{\vartheta\vartheta}=D_{aa}D_{bb} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Hence

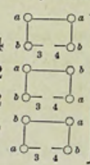
$$\left.\begin{aligned} \vartheta 23 \,.\, \vartheta 32 &= 2D_{aa}D_{bb}-D_{ab}^2, \\ \overline{\vartheta 23} &= \vartheta 23-\tfrac{1}{2}D_{ab}(23), \\ \overline{D_{\vartheta\vartheta}} &= D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2, \end{aligned}\right\} \ldots\ldots\ldots\ldots (18),$$

and

which last agrees with the known formula.

We have next the one-place products of ϑ by itself; namely,

$$\left.\begin{aligned} \vartheta 23.\vartheta 24 &= -D_{aa}D_{bb}(34)=\vartheta 32.\vartheta 42=a12.b13.a52.b54 \\ &= a13.b12.a54.b52 \\ \vartheta 23.\vartheta 42 &= +D_{aa}D_{bb}(34)-D_{ab}.\vartheta 43=a12.b13.a54.b52 \\ \vartheta 32.\vartheta 24 &= +D_{aa}D_{ab}(34)-D_{ab}.\vartheta 34=a13.b12.a52.b54 \end{aligned}\right\} (19);$$



but it is important also to calculate these for the mean value $\overline{\vartheta}$; namely, we have

$$\overline{\vartheta 23} \,.\, \overline{\vartheta 24}=\{\vartheta 23-\tfrac{1}{2}D_{ab}\,(23)\}\,\{\vartheta 24-\tfrac{1}{2}D_{ab}\,(24)\}$$
$$=-D_{aa}D_{bb}\,(34)+\tfrac{1}{2}D_{ab}\,(\vartheta 43+\vartheta 34)-\tfrac{1}{4}D_{ab}^2\,(34)$$
$$=-(D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2)\,(34)=-\overline{D_{\vartheta\vartheta}} \,.\, (34),$$

as it ought to be, confirming equation (18).

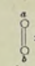
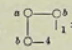
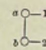
Lastly, from (10) and (11), we get

$$\vartheta 12 \,.\, \vartheta 34+\vartheta 13 \,.\, \vartheta 24=\{D_{aa} \,.\, D_{bb} \,.\, (14)-D_{ab} \,.\, \vartheta 14\}\,(23)$$
$$\vartheta 12 \,.\, \vartheta 34-\vartheta 21 \,.\, \vartheta 43=D_{ab}\,\{23 \,.\, (14)-\vartheta 14\,(23)\}$$
$$=D_{ab}\,\{\vartheta 41 \,.\, (23)-\vartheta 32 \,.\, (14)\} \ldots\ldots (20).$$

There are four one-place products of a, a, b; namely,

$$\left.\begin{aligned} &a12.a32.b34=D_{aa}.b14, \qquad a12 \,.\, a32 \,.\, b43=D_{aa} \,.\, b41, \\ &a12.a23.b34=-D_{aa}.b14+s_a.\vartheta 14+s_a^2.b14=(s_a^2-D_{aa})b14+s_a.\vartheta 14, \\ &a12.a23.b43=-D_{aa}.b41-s_a.\vartheta 14-s_as_b.a41-s_a^2b14+s_as_b(14) \\ &\qquad =(s_a^2-D_{aa}) \,.\, b41-s_a \,.\, \vartheta 14-s_as_b \,.\, a41, \end{aligned}\right\} (21).$$

When the forms are symmetrical, all these reduce to

 $=\pm 2D_{ab}$  $=\pm D_{aa} \,.\, b14$  $=\pm\vartheta 12.$

Three forms a, b, c give rise to the new invariant

$$\begin{aligned} R_{abc}=a12 \,.\, b23 \,.\, c31 &= +a_{11}\,(b_{21}c_{22}-b_{22}c_{12}) \\ &\quad -a_{12}\,(b_{11}c_{22}-b_{12}c_{12}) \\ &\quad -a_{21}\,(b_{21}c_{21}-b_{22}c_{11}) \\ &\quad +a_{22}\,(b_{11}c_{21}-b_{12}c_{11}) \ldots\ldots\ldots\ldots (22), \end{aligned}$$

and the remaining closed products are of the type

$$a21 \,.\, b23 \,.\, c31=\{a12+s_a\,(12)\}\,b23 \,.\, c31=-R_{abc}+s_aD_{bc}-s_as_bs_c.$$

R calculated for the symmetrical functions $\bar{a}$, $\bar{b}$, $\bar{c}$ is

$$\overline{R_{abc}}=\{a12-\tfrac{1}{2}s_a\,(12)\}\,\{b23-\tfrac{1}{2}s_b\,(23)\}\,\{c31-\tfrac{1}{2}s_c\,(31)\}$$
$$=R_{abc}+\tfrac{1}{2}\,(s_aD_{bc}+s_bD_{ca}+s_cD_{ab})-\tfrac{1}{2}s_as_bs_c \ldots\ldots\ldots\ldots (23).$$

It may be mentioned in this place that the determinant made with the coefficients of four forms a, b, c, d has the value

$$\text{Det. } (a, b, c, d)=s_aR_{bcd}-s_bR_{cda}+s_cR_{dab}-s_dR_{abc}=s_a\overline{R_{bcd}}-\&c.,$$

and that no additional invariants are introduced by a greater number of forms.

We now consider the open one-place products of 3 forms. There are evidently four such products having b in the middle; and they are in the first place connected by the formulæ

$$a12 \,.\, b23 \,.\, c34+a42 \,.\, b23 \,.\, c31=-R_{abc}\,(14),$$
$$a12 \,.\, b23 \,.\, c34+a12 \,.\, b32 \,.\, c34=-s_ba13c34=s_b\vartheta_{ac}14+s_as_b \,.\, c14,$$
$$a12 \,.\, b23 \,.\, c34-a34 \,.\, b23 \,.\, c12=-c32\,a34 \,.\, b21+c23 \,.\, a21 \,.\, b43.$$

[Here the paper abruptly ends. It has been found very difficult to devise a satisfactory way of dealing with the other notes on *graphs* on account of the fragmentary condition in which they were left. Prof. H. J. S. Smith says 'the papers tied up with the fragment on *Quadric forms* are little more than mere jottings. They serve to show that Clifford had applied the method to the Quintic and Sextic. They should be preserved....If the letters by which Clifford denotes certain concomitants are those used in Clebsch's *Binäre Formen*, or in any other accessible work or memoir, it might be possible to rescue the pages relating to the Quintic and Sextic.' By the advice, and at the expense, of Dr Spottiswoode, these fragments have been lithographed, and fifty copies have been printed off for circulation among the principal libraries.]

XXXII.

ON MR SPOTTISWOODE'S CONTACT PROBLEMS*.

THE present communication consists of two parts.

The first part treats of the contact of conics with a given surface at a given point; this class of questions was first treated by Mr Spottiswoode in his paper "On the Contact of Conics with Surfaces," and general formulæ applicable to all such questions were given.

The results of that paper are here reproduced with some additions; with the exception of a few collateral theorems, these are all contained in the following Table:—

†Number of five-point conics through fixed point.......... = 6
†Order of surface formed by five-point conics through fixed axis .... = 8
Number of six-point conics through fixed axis............ = 9‡
†Number of seven-point conics.................................. = 70

[* From the *Philosophical Transactions* of the Royal Society of London, Vol. CLXIV. Part 2.]

† These results constitute the additions.

‡ [In the Memoir quoted by Professor Clifford, it was stated that the number of conics passing through a given axis and having six-pointic contact with a surface at a given point is ten. In making this statement I overlooked the fact that, in order to put in evidence that a certain quantity was a factor of the equation which determines the positions of the planes of the conics, the equation was multiplied by a quantity D which is a linear function of the position. In reckoning the degree of the equation this factor must of course be discarded. The degree is consequently less by unity than that stated in the Memoir; viz. it is 9, as proved by Professor Clifford.—Sp. July 3, 1873.]

The second part treats of the contact of a quadric surface with a surface of the order n; and in particular it determines the number of points at which a quadric (other than the tangent plane reckoned twice) can have four-branch contact with the surface. In his paper "On the Contact of Surfaces," Mr Spottiswoode proves that at an arbitrary point on a surface there is no other solution than the doubled tangent plane, and gives the conditions that must be satisfied by those points at which another solution is possible.

The method here adopted is an extension of that applied by Joachimstal to the contact of lines with curves and surfaces. The co-ordinates of a point on a conic are expressed in terms of a single parameter, those of a point on a quadric by two parameters. To determine the intersection with a given surface we have an equation in the parameter or parameters, and the conditions of contact are expressed in terms of the coefficients of that equation. The special case of the intersection of a quadric with a cubic surface is treated by the method of representation on a plane.

PART I.—THE CONTACT OF CONICS WITH SURFACES OF ORDER n.

I.

The current plane-coordinates being denoted by

$$X_1,\ X_2,\ X_3,\ X_4,$$

let the equations of the three points A, B, C be respectively

$$0 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 \equiv \Sigma aX,$$
$$0 = \Sigma bX,$$
$$0 = \Sigma cX.$$

The quantities a_i, b_i, c_i $(i=1, 2, 3, 4)$ are the coordinates of the points A, B, C. The symbol A itself I shall use indifferently, as denoting either the form ΣaX or the differential operator

$$a_1\partial_{x_1} + a_2\partial_{x_2} + a_3\partial_{x_3} + a_4\partial_{x_4} \equiv \Sigma a\partial_x,$$

where x_1, x_2, x_3, x_4 are the current point-coordinates. It will be seen in the sequel that this double meaning is useful, while it does not introduce any confusion. Similar interpretations are of course to be given to the symbols B, C, and the like.

Consider now the point

$$P \equiv A + \theta B + \theta^2 C,$$

whose coordinates are $a_i + \theta b_i + \theta^2 c_i$ $(i = 1, 2, 3, 4)$. If we suppose θ to take all possible values, the point P will describe a conic section whose tangential equation is

$$0 = 4AC - B^2 \equiv K_2.$$

To the value $\theta = 0$ corresponds the point A, to $\theta = \infty$ the point C; while the equation shews that B is the intersection of the tangents at A and C. (Fig. 51.)

To find the point at which this conic intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation in θ of the order $2n$, the solution of which will give the values of θ belonging to the $2n$ points of intersection.

If in this equation the term independent of θ vanishes, then $\theta = 0$ is a root of the equation; consequently the point A is one of the intersections, or the surface u_n passes through the point A. If also the coefficient of θ vanishes, another root of the equation coincides with zero, and *two* points of intersection are at A. And generally if the coefficient of θ^m is the first that does not vanish, m roots of the equation coincide with zero, and m points of intersection are united at A.

The result of substituting the coordinates of any point P in u_n may be conveniently represented by means of the differential operator P. It is known, in fact, that

$$(p_1\partial_{x_1} + p_2\partial_{x_2} + p_3\partial_{x_3} + p_4\partial_{x_4})^n \cdot (x_1, x_2, x_3, x_4)^n \equiv \underline{|n} \cdot (p_1, p_2, p_3, p_4)^n;$$

or, which is the same thing, $P^n u_n$ is $\underline{|n}$ times the result of substituting the coordinates of P in u_n. Our result may therefore be stated in the following form :—

The necessary and sufficient conditions that the conic $K_2 \equiv 4AC - B^2$ *may have* m*-point contact with the surface* u_n

at the point A, *are that in the expansion of* $(A+\theta B+\theta^2 C)^n . u_n$ *in powers of* θ, *the* m*th power of* θ *is the lowest whose coefficient does not vanish.*

II.

Equating to zero the coefficients of 1, θ, θ^2 in this expansion, we obtain

$$0 = A^n . u_n,$$
$$0 = nA^{n-1} B . u_n,$$
$$0 = \tfrac{1}{2}n(n-1) A^{n-2} B^2 . u_n + nA^{n-1} C . u_n.$$

Before proceeding further with these equations, it is convenient to make the following remarks upon their nature, which will serve to simplify the expression of them.

In the first place, then, we have here a series of relations among the coordinates of the points A, B, C and the coefficients of the surface u_n; and the determination of the coordinates and coefficients so as to satisfy a certain number of the relations presents us with the solution of various geometrical problems. These problems fall naturally into three classes.

1. The surface u_n and the point A are given. In this case the unknowns are the ratios of the eight quantities b, c, a singly infinite number of solutions corresponding to each conic; and we are accordingly able to satisfy seven of the equations*. The problem here is to find the number of conics which have seven-point contact at a given point of a given surface.

We may, however, impose beforehand certain restrictions upon the values of the unknowns, and so consider problems which involve a less number of the equations. While the number of the septactic conics is definite, the sextactic conics form a singly infinite series; and we may ask what is the number of them:

(*a*) whose planes pass through a given point,

(*b*) which meet a given line, or

(*c*) which touch a given plane.

* Viz. six besides the first, which is satisfied identically.

The quintactic conics, again, form a doubly infinite system, and we may inquire about the number of them which satisfy two conditions; *e.g.* which pass through a given point.

2. The surface u_n is given, but not the point A. In this case, as the point A is only restricted to be a point on the surface, we have two more unknowns, making nine in all. The problems here are, to find the order of the curve on the surface at every point of which there is a conic having eight-point contact, and to find the number of points at which there is a conic having nine-point contact.

As before, however, there are certain derived problems coming under this head which involve a less number of equations. We may seek the order of the curve traced out by points of contact of septactic conics satisfying one condition, sextactic satisfying two, &c.; or we may seek the number of septactic conics satisfying two conditions, sextactic satisfying three, &c.

3. The surface is not wholly given. We may here assign a number of relations sufficient to eliminate the quantities a, b, c, leaving one or more relations among the coefficients of u_n. The problems here are such as:—to find the number of surfaces in a pencil $u_n + \lambda v_n$ which admit of ten-point contact with some conic, or one of whose nine-point conics meets a given line.

In the present communication only problems of class 1 will be considered; the formulæ in this case may be very considerably simplified. The quantities a and the coefficients of u_n, then, are data of the problem; so that the first of our equations, $A^n u_n = 0$, is satisfied by hypothesis. The next equation, $A^{n-1} Bu_n = 0$, signifies that the point B lies in the tangent plane at A, as it obviously must if the conic touch the surface at A. We shall suppose this also to be satisfied from the commencement; that is, we shall regard B as a point moving in the tangent plane, and to be determined by construction in that plane. This may be effected analytically if we substitute for B, $\lambda A + \mu B + \nu B'$, where now B, B' are regarded as fixed points in the tangent plane, and the three unknowns λ, μ, ν take the place of the four quantities b. There is, however, as will be seen, no occasion to make the substitution explicitly.

This being so, any relation involving B only beside the data must be regarded as the equation of a curve in the tangent plane. For example, $A^{n-2} B^2 u_n = 0$, expressing that B lies on the quadric polar of A, is the equation of the two chief tangents at that point. Generally, $B^n u_n = 0$ is the equation of the intersection with u_n of the tangent plane; and the curves $AB^{n-1}u_n = 0$, $A^2B^{n-2}u_n = 0$, &c. are the successive polars of A in regard to that intersection.

The terms entering into our equations are of the general form

$$\theta^{p+2q} \cdot \frac{\underline{|n}}{\underline{|n-p-q}\ \underline{|p}\ \underline{|q}} A^{n-p-q} B^p C^q \cdot u_n.$$

Any term, therefore, is completely determined by the two numbers p and q, and might, for any thing that has yet appeared, be denoted by (p, q). In view, however, of subsequent substitutions, we shall keep in evidence the manner in which B and C are involved, and denote the term in question by the symbol $\theta^{p+2q}(B^p C^q)$.

As we do not consider any higher than seven-point contact, we have only the five equations:

$$0 = (B^2) + (C) \quad \ldots\ldots (3),$$
$$0 = (B^3) + (BC) \quad \ldots\ldots (4),$$
$$0 = (B^4) + (B^2C) + (C^2) \quad \ldots\ldots (5),$$
$$0 = (B^5) + (B^3C) + (BC^2) \quad \ldots\ldots (6),$$
$$0 = (B^6) + (B^4C) + (B^2C^2) + (C^3) \quad \ldots\ldots (7).$$

III. *Conics through a fixed point.*

Combining equation (3) successively with (4) and (5), so as to obtain results homogeneous in B, we find

$$0 = (B^3)(C) - (BC)(B^2),$$
$$0 = (B^4)(C)^2 - (B^2C)(B^2)(C) + (C^2)(B^2)^2.$$

If we regard C as a fixed point, these are equations of a cubic and a quartic curve in the tangent plane, on each of which B must lie if the conic K_2 has five-point contact. But these curves have a common node at A, and common tangents at it; for every term in each has at least one factor of the form (B^m), which we know to represent a polar of A in regard to the intersection of u_n by the tangent plane—that is, a curve touched by the chief tangents at A. Of their 12 intersections, then, 6 coincide with the point A; and there remain *six conics having five-point contact at* A *which pass through an arbitrary point* C.

IV. *Conics meeting a fixed axis through* A.

If the point C, instead of being altogether given, is movable on a fixed straight line through A, we may represent it by $A + \lambda C$; where now C is really a fixed point, and λ a quantity to be determined. When we make this substitution in our equations, they become*

$$0 = (B^2) + \lambda (C),$$
$$0 = (B^3) + \lambda (BC),$$
$$0 = (B^4) + (B^2) + \lambda (B^2C) + 2\lambda (C) + \lambda^2 (C^2),$$
$$0 = (B^5) + (B^3) + \lambda (B^3C) + 2\lambda (BC) + \lambda^2 (BC^2),$$
$$0 = (B^6) + (B^4) + \lambda (B^4C) + (B^2) + 2\lambda (B^2C) + \lambda^2 (B^2C^2) + 3\lambda (C) + 3\lambda^2 (C^2) + \lambda^3 (C^3).$$

The last three admit of obvious simplifications by aid of the previous ones; and the system may finally be written

$$0 = (B^2) + \lambda (C) \quad \ldots\ldots (3'),$$
$$0 = (B^3) + \lambda (BC) \quad \ldots\ldots (4'),$$
$$(B^2) = (B^4) + \lambda (B^2C) + \lambda^2 (C^2) \quad \ldots\ldots (5'),$$
$$(B^3) = (B^5) + \lambda (B^3C) + \lambda^2 (BC^2) \quad \ldots\ldots (6'),$$
$$(B^4) - \lambda^2 (C^2) = (B^6) + \lambda (B^4C) + \lambda^2 (B^2C^2) + \lambda^3 (C^2) \quad \ldots\ldots (7').$$

* It must be remembered that (B) and terms involving A only vanish by hypothesis. The formulæ have also been simplified by the omission of certain coefficients depending on n which do not affect the final results.

Locus of poles of axis in regard to four-point conics.

If we select any plane through the line AC, there will be a singly infinite number of conics in the plane having four-point contact with the surface at A. The line AC will, as is well known, have the same pole in regard to all these conics—that is to say, the point B will be the same for the whole system. If we now allow the plane to turn round the axis, the point B will trace out a curve in the tangent plane. The equation to this curve is got by eliminating λ between equations (3′) and (4′); namely, it is

$$0 = (B^2)(BC) - (B^3)(C) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4'').$$

We see, therefore, that *the locus of the poles of an axis in regard to all the four-point conics whose planes pass through it is a cubic curve in the tangent plane touching the chief tangents at* A, *which point is therefore a node on the curve.*

We might have inferred this from the fact that on any line through A there is only one point B, while this point coincides with A in the case of the two chief tangents; since at a point of inflexion all the four-point conics contain the inflexional tangent.

Number of six-point conics through an axis.

We have now to determine B so that the equations (3′), (4′), (5′), (6′) may be simultaneously satisfied. We know already that B must lie on the cubic (4″); it is necessary therefore to find some other locus on which it has to lie. First of all, then, we must eliminate λ between (3′) and (5′) and between (3′) and (6′); the results are,

$$(B^2)(C)^2 = (B^4)(C)^2 - (B^2C)(B^2)(C) + (B^2)^2(C^2) \equiv p_4, \text{ say};$$
$$(B^3)(C)^2 = (B^5)(C)^2 - (B^3C)(B^2)(C) + (B^2)^2(BC^2) \equiv q_5, \text{ say}.$$

Here $p_4 = 0$ and $q_5 = 0$ are curves touching the chief tangents at A, and of the degrees four and five respectively. But the equations are not homogeneous; in fact only the ratios and not the absolute values of the quantities a were determined by the

fixing of the point A, and they may be regarded as involving an arbitrary factor whose square affects the left-hand side of the equations. It is, however, at once eliminated, and we obtain the homogeneous result,

$$(B^2) \cdot q_5 = (B^3) \cdot p_4.$$

This is a curve of order 7 having two branches in each of the chief directions at A. Of its 21 intersections with the cubic (4″), then, 12 coincide with A, and there remain *nine positions of* B *which give sextactic conics through the fixed axis;* or we may say, *of the sextactic conics at the point* A, *there are nine whose planes pass through an arbitrary point* C.

System of five-point conics through an axis.

Since there is one five-point conic in every plane, if we consider all the planes through a fixed axis we shall obtain a singly infinite number of five-point conics. Of this system there is only one conic whose plane passes through an arbitrary point, viz. the conic determined by the plane through that point and the axis.

There are eight conics of the system which meet an arbitrary line.

The number of conics which meet an arbitrary line is clearly the same as the order of the surface which they trace out. Now, since through every point on the axis can be drawn six conics of the system (as proved in the last section), the axis is a six-fold line on the surface. The section of the surface, then, by a plane through the axis is made up of the axis taken six times over and the conic in that plane; or it is of the order *eight*. Q. E. D.

V. *Conics not subject to any condition.*

In order to get rid of the restriction of meeting a fixed axis, we must again modify our fundamental equations. We have to put them into a form in which they will represent *any* conic touching the surface u_n at the point A. For this purpose it is necessary and sufficient that C should be movable over a plane passing through A; since every conic through A must cut the plane in one other point, but this may be any point of the plane. We attain this analytically by substituting for λC in the second set of equations, $\lambda C + \mu D$, where C and D are now fixed points not in the tangent plane and not in any straight line through A. This is equivalent to still considering the conics which meet a given axis, but allowing that axis to move over a fixed plane.

The transformed equations are :—

$$0 = (B^2) + \lambda (C) + \mu (D) \quad \ldots\ldots (3'''),$$

$$0 = (B^3) + \lambda (BC) + \mu (BD) \quad \ldots\ldots (4'''),$$

$$(B^2) = (B^4) + \lambda (B^2C) + \mu (B^2D) + \lambda^2 (C^2) + 2\lambda\mu (CD) + \mu^2 (D^2) \quad (5'''),$$

$$(B^3) = (B^5) + \lambda (B^3C) + \mu (B^3D) + \lambda^2 (BC^2) + 2\lambda\mu (BCD) + \mu^2 (BD^2) \quad \ldots\ldots (6'''),$$

$$\left. \begin{aligned} (B^4) - \lambda^2 (C^2) - 2\lambda\mu (CD) - \mu^2 (D^2) = (B^6) + \lambda (B^4C) \\ + \mu (B^4D) + \lambda^2 (B^2C^2) + 2\lambda\mu (B^2CD) + \mu^2 (B^2D^2) \\ + \lambda^3 (C^3) + 3\lambda^2\mu (C^2D) + 3\lambda\mu^2 (CD^2) + \mu^3 (D^3) \end{aligned} \right\} \ldots\ldots (7''').$$

Locus of poles of axis in given plane in regard to sextactics.

From the first two of these equations we obtain

$$1 : \lambda : \mu = (BD)(C) - (BC)(D) : (B^3)(D) - (B^2)(BD) : (B^3)(C) - (B^2)(BC) = n_1 : l_3 : m_3, \text{ say};$$

here $n_1 = 0$ is the equation to a straight line passing through A, while $l_3 = 0$, $m_3 = 0$ are cubics touching the chief tangents at A.

Substituting these values in (5‴) and (6‴), we obtain

$$n_1^2 (B^2) = n_1^2 (B^4) + n_1 l_3 (B^2C) + n_1 m_3 (B^2D) + l_3^2 (C^2) + 2 l_3 m_3 (CD) + m_3^2 (D^2) = u_6, \text{ say};$$

$$n_1^2 (B^3) = n_1^2 (B^5) + n_1 l_3 (B^3C) + n_1 m_3 (B^3D) + l_3^2 (BC^2) + 2 l_3 m_3 (BCD) + m_3^2 (BD^2) = v_7, \text{ say}.$$

Here the curves $u_6 = 0$, $v_7 = 0$ are of the sixth and seventh orders respectively, each of them having *one* branch at A in each of the chief directions, and one other branch different for the two curves.

The equations

$$n_1^2 (B^2) = u_6,$$

$$n_1^2 (B^3) = v_7$$

must hold for six-point contact, but they are not homogeneous. Eliminating n_1^2, however, there results

$$(B^2) \,.\, v_7 = (B^3) \,.\, u_6,$$

a curve of the ninth order, locus of the poles in regard to the sextactic conics of an axis moving in a fixed plane. This curve has *two* branches at A in each of the chief directions, and *one* other branch; and consequently is met by the plane ACD in *five* points coinciding with A, and in *four* other points. Now the plane ACD does not in general contain a sextactic conic; the pole of the axis can therefore only be in this plane when the axis itself is in the tangent plane. In this case there is a certain number of proper sextactic conics in planes through the axis, and it is clear that the pole of the axis in regard to any such conic is the point A. These conics, therefore, correspond to the *five* intersections of the plane ACD with the locus of poles which coincide with A; or *through an axis in the tangent plane can be drawn five proper sextactic conics.* In the tangent plane itself there are four improper conics having six-point contact; viz. the pair of chief tangents, which (as a sharp conic or line-pair reckoned among conics given tangentially) counts for two, and each chief tangent doubled.

Number of septactic conics.

There is a finite number of septactic conics at the point A; each of these meets the plane ACD in a determinate point $A + \lambda C + \mu D$, and fixes thereby a position of the point B. These positions of the point B must necessarily lie in the 9thic locus just investigated; it remains only to find a homogeneous relation which shall determine another locus for B, and to count the number of their intersections.

To this end we must first substitute for $1 : \lambda : \mu$ their values $n_1 : l_3 : m_3$ in equation (7‴). The result is

$$n_1^3(B^4) - n_1 l_3^2 (C^2) - 2n_1 l_3 m_3 (CD) - n_1 m_3^2 (D^2)$$
$$= n_1^3 (B^5) + n_1^2 l_3 (B^4 C) + n_1 m_3 (B^4 D)$$
$$+ n_1 l_3^2 (B^2 C^2) + 2n_1 l_3 m_3 (B^2 CD) + n_1 m_3^2 (B^2 D)$$
$$+ l_2^3 (C^3) + 3 l_3^2 m_3 (C^2 D) + 3 l_3 m_3^2 (CD^2) + m_3^3 (D^3),$$

or

$$n_1 t_5 = w_9, \text{ say.}$$

The curve w_9 has *one* branch at A in each of the chief directions and one other branch. The curve t_6 has one branch in each of the chief directions and *two* other branches.

The equations (5), (6), (7) have now become

$$n_1^2 (B^2) = u_6,$$
$$n_1^2 (B^3) = v_7,$$
$$n_1 t_6 = w_9.$$

The first two of these give us the curve already considered,

$$(B^2) . v_7 = (B^3) . u_7,$$

which has at A two branches in the chief directions and one other. The first and third give

$$n_1 . (B^2) . w_9 t_6 u_6,$$

a curve of order 12, having two branches in each of the chief directions and two other branches. Of the 108 intersections of these curves, then, $24 + 8 + 4 + 2 = 38$ coincide with A, leaving 70 for the number of septactic conics.

PART II. THE CONTACT OF QUADRIC SURFACES WITH SURFACES OF ORDER n.

I. *Conditions of contact.*

Let A, B, C, D be four points forming a tetrahedron, whose tangential equations are

$$0 = \Sigma aX, \ \Sigma bX, \ \Sigma cX, \ \Sigma dX$$

respectively, their coordinates being a_i, b_i, c_i, d_i $(i = 1, 2, 3, 4)$. Then the point

$$P \equiv A + \theta B + \phi C + \theta\phi D,$$

whose coordinates are

$$p_i \equiv a_i + \theta b_i + \phi c_i + \theta\phi d_i \ (i = 1, 2, 3, 4),$$

will, if we suppose θ and ϕ to take all possible values, trace out a quadric surface whose tangential equation is

$$0 = AD - BC \equiv Q_2.$$

To the pair of values

$\theta = 0$, $\phi = 0$, corresponds the point A,
$\theta = \infty$, $\phi = 0$, " " B,
$\theta = 0$, $\phi = \infty$, " " C,
$\theta = \infty$, $\phi = \infty$, " " D.

The equation shews that AB, AC, BD, CD are generating lines.

If, now, we wish to find the nature of the curve in which this quadric intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation which is of the order n in θ and in ϕ separately. If we regard θ and ϕ as coordinates of a point on the quadric surface, the equation just found is that of the curve of intersection.

Suppose that in this equation the term independent of θ and ϕ vanishes, then the equation is satisfied by the pair of values $\theta = 0$, $\phi = 0$, or the curve of intersection passes through the point A. Now the various directions in which we may start from the point A are determined by the initial value of the ratio $\theta : \phi$ when we move along them. The direction in which the intersection-curve starts from A is therefore that obtained by neglecting in the equation terms of higher order than the first; and we see that there is only one such direction.

If, however, not only the constant term but the coefficients of θ and ϕ in the equation vanish, the initial directions are obtained by equating to zero the terms of the second order, *i.e.* by neglecting in the equation all terms of a higher order than the second. In this case, then, there are two such directions, the intersection-curve has a double point at A, and the quadric has with the surface u_n an ordinary or *two-branch* contact.

Again, if the coefficients of the terms of the third order are the first that do not vanish, the initial directions are obtained by neglecting all the terms of higher order, and there are consequently three of them. Thus the intersection-curve has a *triple* point at A, and the two surfaces have a *three-branch* contact.

And so generally, if the coefficients of the terms of the m^{th} order are the first that do not vanish, the intersection-curve has at A a multiple point of order m, and the two surfaces have at that point an m-branch contact.

The result of these considerations may be stated as follows:

The necessary and sufficient conditions that the quadric $Q_2 \equiv AD - BC$ *may have* m*-branch contact with the surface* u_n *at the point* A, *are that in the expansion of*

$$(A + \theta B + \phi C + \theta\phi D)^n \cdot u_n$$

in powers and products of θ *and* ϕ, *the terms of order* m *in* θ *and* ϕ *are the lowest whose coefficients do not vanish.*

II. *Quadrics of four-branch contact.*

The equations we shall have to employ are so simple in form that it is unnecessary to employ the abridged notation of the former Part. We shall merely omit the operand u_n, and any common factor of the binomial coefficients.

The conditions for ordinary contact are then

$$0 = A^n, \quad 0 = A^{n-1}B, \quad 0 = A^{n-1}C.$$

The first of these expresses that A is a point on the surface u_n, the second and third that B and C are on the tangent plane at A.

The further conditions for three-branch contact are

$$0 = A^{n-2}B^2, \quad 0 = A^{n-2}C^2, \quad 0 = A^{n-1}D + (n-1)A^{n-2}BC.$$

The first two of these shew that B and C are points on the chief tangents at A. If we regard the absolute values of the coordinates of A as given, then it appears from the third equation that B and C may be chosen arbitrarily on the chief tangents and D anywhere in space, the equation giving the relation between the absolute values of their coordinates which determines the particular surface $AD - BC = 0$.

For four-branch contact we have the additional equations,

$$0 = A^{n-3}B^3, \qquad 0 = A^{n-3}C^3,$$
$$0 = 2A^{n-2}BD + (n-2)A^{n-3}B^2C,$$
$$0 = 2A^{n-2}CD + (n-2)A^{n-3}BC^2.$$

The first two of these indicate that B and C lie on the polar cubic of A in regard to the section u_n by the tangent plane. Now this polar cubic has a node at A, whose tangents are the chief tangents. Each of these lines therefore meets the cubic in three points at A, and cannot have any other point on the curve unless it be itself a part of the cubic. But the points B and C have to lie one on each of the chief tangents. In order, therefore, that all the equations may be satisfied, the polar cubic in question must break up into the two chief tangents and some other line.

This condition may be put into another form. For if we seek the points in which the line AB meets the surface, by substituting the coordinates of $A + \lambda B$ in $u_n = 0$, the conditions $A^n u_n = 0$, $A^{n-1}Bu_n = 0$, $A^{n-2}B^2u_n = 0$, $A^{n-3}B^3u_n = 0$ indicate that *four* roots of the equation are equal to zero, or that the line meets the surface in four consecutive points at A. We find, therefore, that

Those points of a surface at which a quadric may have four-branched contact are the points at which each chief tangent meets the surface in four consecutive points, or, which is the same thing, *the points whose polar cubic contains the chief tangent.*

The number of these points has been counted by Clebsch, *Crelle*, LXIII. 14*, and turns out to be

$$n(41n^2 - 162n + 162).$$

A point of this nature being given, one quadric surface having four-branch contact at it may be drawn through another arbitrary point.

The coordinates of the point A being given as to their absolute values, let us substitute for B and C, $A + \lambda B$, $A + \mu C$; where now B and C are fixed points on the chief tangents, whose coordinates are given absolutely. This being so, the following equations are satisfied *ex hypothesi*:—

$$A^n u_n = 0, \quad A^{n-1} B u_n = 0, \quad A^{n-1} C u_n = 0,$$
$$A^{n-2} B^2 u_n = 0, \quad A^{n-2} C^2 u_n = 0, \quad A^{n-3} B^3 u_n = 0, \quad A^{n-3} C^3 u_n = 0;$$

from which it follows at once, for example, that

$$A^{n-3}(A + \lambda B)^3 u_n = 0.$$

For D also let us substitute νD, where the coordinates of D are now given absolutely. Our three remaining equations are (omitting for shortness the mention of A)

$$0 = \nu D + (n-1)\lambda\mu \,.\, BC \quad \ldots\ldots (1),$$
$$0 = \nu D + (n-2)\lambda\mu \,.\, BC + \nu\lambda \,.\, BD + \tfrac{1}{2}(n-2)\lambda^2\mu \,.\, B^2C \ldots (2),$$
$$0 = \nu D + (n-2)\lambda\mu \,.\, BC + \nu\mu \,.\, CD + \tfrac{1}{2}(n-2)\lambda\mu^2 \,.\, BC^2 \ldots (3);$$

and it remains only to shew that these equations determine uniquely λ, μ, ν.

If we subtract (1) from (2) and (3) successively, we obtain

$$0 = -\mu \,.\, BC + \nu \,.\, BD + \tfrac{1}{2}(n-2)\lambda\mu \,.\, B^2C \ldots\ldots (4),$$
$$0 = -\lambda \,.\, BC + \nu \,.\, CD + \tfrac{1}{2}(n-2)\lambda\mu \,.\, BC^2 \ldots\ldots (5),$$

the values $\lambda = 0$, $\mu = 0$ not being admissible. But if we substitute in (4) and (5) the value of $\lambda\mu$ derived from (1), we obtain two simple equations which determine the ratios $\lambda : \mu : \nu$; after which the absolute values are uniquely determined from (1).

* The investigation is given by Salmon, *Geom. Three Dim.* p. 444.

It is otherwise obvious that if $0 = q_2$ be the point-equation to a four-branch quadric, and $0 = t_1$ to the tangent plane, $0 = q_2 + \rho t_1^2$ will be the equation to a pencil of quadrics having four-branch contact, one of which may be made to pass through an arbitrary point.

Special investigation for n = 3.

In the case in which u_n is a cubic surface, the points in question are clearly the 135 points of contact of the 45 triple tangent planes, namely, the intersections of the 27 lines. This case may be conveniently studied by means of the representation of that surface on a plane. The plane sections of the surface here correspond to a system of cubics having six common points; the quadric sections therefore to sextics having nodes at these points. The problem is then to draw a sextic curve having six given nodes and a quadruple point elsewhere.

The twenty-seven lines of the cubic are represented by

the 6 fixed points,
the 15 lines joining them, and
the 6 conics each through five of them.

It shall now be proved that the sextic must include two of these; and that for each pair that meet there is a singly infinite number of sextics.

The sextic cannot be a proper curve; for six nodes and a quadruple point are equivalent to twelve nodes, and a proper sextic can have only ten. Nevertheless we may apply a quadric transformation to it, whose principal points are the quadruple point and two of the nodes. The sextic is thus reduced to a quartic, passing 2, 0, 0 times through the principal points respectively and having four other nodes. But a quartic having five nodes must be made up of a conic and two straight lines. Now, if the node at a principal point is the intersection of the two lines, the original sextic was made up of two lines, passing each through two of the six points, and a quartic having nodes at their intersection and the remaining two points and passing through those four. Let a, b, c, d, e, f be the six points, p the intersection of ab, cd; then the sextic is in this case made up of

lines ab, cd,

quartic $p^2e^2f^2abcd$.

If, however, in the transformed figure the node at a principal point was the intersection of a line with the conic, the original sextic was made up of a line, a conic, and a nodal cubic, viz. if p be the intersection of the line ab and the conic $bcdef$, the nodal cubic is p^2acdef.

Now we know that we can draw a singly infinite number of quartics with three fixed nodes and four fixed points, or of cubics with a fixed node and five fixed points. Hence in both these cases the sextic includes two representatives of straight lines in the cubic, together with another curve which may be arbitrarily chosen from a pencil.

XXXIII.

ON THE CLASSIFICATION OF LOCI*.

PART I. CURVES.

By a *curve* we mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and then, if these be connected by $k-1$ equations of any order, the curve is either the whole aggregate of common solutions of these equations, or, when this breaks up into algebraically distinct parts, the curve is one of these parts. It is thus convenient to employ still further the language of geometry, and to speak of such a curve as the complete or partial intersection of $k-1$ loci in flat space of k dimensions, or, as we shall sometimes say, in a k-flat. If a certain number, say h, of the equations are linear, it is evidently possible by a linear transformation to make these equations equate h of the coordinates to zero; it is then convenient to leave these coordinates out of consideration altogether, and only to regard the remaining $k-h-1$ equations between $k-h$ coordinates. In this case the curve will, therefore, be regarded as a curve in flat space of $k-h$ dimensions. And, in general, when we speak of a curve as in flat space of k dimensions, we mean that it cannot exist in flat space of $k-1$ dimensions.

The whole aggregate of linear complexes may be regarded

[* From the *Philosophical Transactions of the Royal Society*.—Part II. 1878, pp. 663—681.]

as constituting a space of five dimensions, in which the *special* complexes, or straight lines, constitute a quadric locus. A ruled surface, or scroll, will be thus regarded as a curve lying in a quadric locus in a flat space of five dimensions. If, however, the generators of the scroll all belong to the same linear complex, the scroll must be regarded as a curve lying in a quadric locus in a flat space of *four* dimensions. And if, further, the scroll has two linear directrices, so that the generators belong to a linear congruence, then the scroll may be regarded as a curve lying on an ordinary quadric surface in three dimensions. Thus, for example, quartic scrolls having two linear directrices correspond either to quadri-quadric curves of deficiency 1 (that is, they are *elliptic* curves whose coordinates may be expressed as elliptic functions of one variable), or to the curves of deficiency 0 which are the partial intersections of a quadric and a cubic surface (that is, they are unicursal curves).

This view of ruled surfaces is made excellent use of by Voss, "Zur Theorie der windschiefen Flächen," *Math. Annalen*, Vol. VIII. p. 54.

Similar considerations apply to surfaces. By a *surface* we shall mean, in general, a continuous two-dimensional aggregate (which may also be called a *two-spread* or *two-way locus*) of any elements whatever, curves, surfaces, complexes, &c., defined by the whole or a portion of the system of solutions of $k-2$ equations among k coordinates. We shall assume that none of these equations are linear, and then shall speak of the surface as in a flat space of k dimensions. We shall in certain cases go further, and speak of an h-spread or h-way locus, viz., a locus determined by the whole or an algebraically separate portion of the system of solutions of $k-h$ equations among k coordinates; if none of these equations are linear, the h-way locus will be said to be in k dimensions. The general point of view is that of Professor Cayley, "On the Curves which satisfy given Conditions," *Phil. Trans.*, Vol. 158 (1868), pp. 75—144; the methods of enumeration are those of Dr Salmon, "Solid Geometry," p. 261.

THEOREM A. *Every proper curve of the* n[th] *order is in a flat space of* n *dimensions or less.* For through $n+1$ points of it we can draw a flat space of n dimensions, which must therefore contain the curve, since it meets it in a number of points greater than its order.

Thus, for example, there is no curve of the second order, in space of any number of dimensions, except a plane conic. If, therefore a system of curves, in a plane or on any surface, is such that two curves of the system can be drawn through an arbitrary point, then the coordinates of a varying curve of the system may be represented by $x_i + 2\theta y_i + \theta^2 z_i$ $(i = 1, 2, 3 \ldots k)$, and the envelope of the system is, in the case of plane curves, a curve having the equation $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, where U, V, W are three curves of the system; in the case of curves on a surface, it is the intersection of the surface with another having an equation of that form*.

* Professor Henrici has kindly written for me the following notes in elucidation of this argument:—

"In the first sentence of the paper it is stated that by a *curve* is meant any one-dimensional aggregate of any sort of elements. The definitions given are algebraical, but the reasoning later on becomes more and more geometrical.

"In this note the connexion between the algebraical definition and the geometrical reasoning will be shewn in the case where the elements are plane curves of order n.

"If we suppose a curve given by its equation in point coordinates we may take the coefficients as homogeneous coordinates of the curve.

"As there are $\frac{n(n+3)}{2}$ ratios of these coefficients, it follows that all curves of order n in a plane constitute a $\frac{n(n+3)}{2}$ spread, and this will be a *flat* spread as no relation has been supposed between the coordinates.

"To determine in this spread a k-flat, $k < \frac{n(n+3)}{2}$, we have to assume a sufficient number of equations between the coordinates, or denoting by $n_1, n_2, \ldots$ curves of order n we may write down the equation of one element in the k-flat in the form $a_1 u_1 + a_2 u_2 + \ldots + a_{k+1} u_{k+1} = 0$, and take the k ratios of the a as the coordinates of a variable curve.

"For $k=2$ we get a *net* as the flat space of two dimensions or as the *plane* in this space, and for $k=1$ a pencil corresponding to the *line*.

"If, on the other hand, we assume in the k-flat $k-1$ equations between the coordinates a, there remains a singly infinite number of curves, that is according

To particularise still further, a system of conics having the characteristic $\mu = 2$ must always have quadruple contact with a quartic curve; and the different species may be enumerated by studying the successive degeneration of the curve, ending with the fundamental system $\nu = 1$, when it breaks up into four straight lines.

So again, there is no quadric scroll, in any number of dimensions, except the ordinary quadric surface which is in flat space of three dimensions.

A curve of the third order must be either the known skew cubic in three dimensions, or a plane cubic. Hence, if a system

to Professor Clifford a *curve* (with curves as elements), according to the usual nomenclature a series of curves.

"To determine the order of this curve we have to find the number of elements on it which satisfy a linear relation between the coordinates. In our case the condition that a curve shall pass through a given point gives such a relation, and the number of curves through a point is the *order* in question.

"Hence, if we wish to extend a theorem relating to a curve (in the ordinary sense with points as elements, but in any number of dimensions) to a proposition relating to a series of curves, or if we wish to illustrate in a plane a theorem relating to a curve in more dimensions than three, we have instead of a point on the curve to take a curve in the series, and to replace the order of the curve by the index of the series.

"The theorem that every curve of order two is a *plane* curve becomes thus—the curves in a series of index 2 belong to a *net*.

"Further, the coordinates of a point on a conic may be represented as expressions of the second degree in a variable parameter, say by $x_i + 2\theta y_i + \theta^2 z_i$; where $i = 1, 2, 3$, if the coordinates are taken in the plane of the conic, but if they are taken in space we have to take $i = 1, 2, 3, 4$, and so on for more dimensions. The locus of these points, that is, the conic, is then given by an equation of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

where U, V, W are three of the points.

"If we apply this to our series we obtain the results stated in the text, viz., the coordinates of any curve of a series such that two curves pass through a given point are of the form quoted, and the equation of the envelope is of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

U, V, W being three of the curves.

"Similarly, if the series is such that three pass through any point, then the series may be considered as a 'curve' of order three, and the statements made in the text follow at once from the known properties about cubic curves, which are either unicursal (twisted, or plane nodal, cubics) or they are plane curves of deficiency one."—January, 1879.

of curves be such that three of them can be drawn through an arbitrary point, the equation of any curve of the system is of one of the two forms—

$$U + 3Vt + 3Wt^2 + Xt^3 = 0,$$
$$U + V \operatorname{sn}^2 u + 2W \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u = 0,$$

where t, u are parameters. Hence it is easy to write down the equations to the envelopes in the two cases, and to enumerate the distinct species.

A cubic scroll must be of the nature of the skew cubic, because it is a curve (with complexes for elements) which is obliged to lie on a quadric locus (that of the special complexes, or straight lines).

THEOREM B. *A curve of order* n *in flat space of* k *dimensions (and no less) may be represented, point for point, on a curve of order* n − k + 2 *in a plane.*

The proposition is obvious when $k = 3$. The cone standing on a curve of order n (in ordinary space of three dimensions), and having its vertex at a point of the curve, is of order $n - 1$; if then we cut this cone by a plane, we have the tortuous curve represented, point for point, on a plane curve of order $n - 1$.

Now this process is applicable in general. Starting with an arbitrary point, P, of a curve in any number of dimensions, let us join this point to all the other points of the curve; we shall thus get a *cone* of order $n - 1$. For any flat locus of $k - 1$ dimensions drawn through the point P must meet the curve in n points, of which P is one; and therefore it must meet the cone in $n - 1$ lines. Hence, if we cut this cone by such a flat $(k-1)$ way locus *not* passing through P, we shall get a curve of order $n - 1$ in flat space of $k - 1$ dimensions, which is a point-for-point representation of the original curve. By continuing this process we may go on diminishing the order of the curve and the number of dimensions by equal quantities, until we have subtracted $k - 2$ from each; when we are left with a curve of order $n - k + 2$ in a plane.

The reduction may, however, be effected in one step. A flat $(k-2)$ way locus may be drawn through $k-1$ arbitrary points. Suppose it to contain $k-2$ consecutive points of the curve at P, and another variable point, Q, of the curve. Such a locus will meet an arbitrary plane in one point, R. As Q then moves about on the curve, R will trace out on the plane a curve which corresponds to it, point for point. But this curve is of order $n-k+2$, for a flat $(k-1)$ way locus, passing through $k-2$ consecutive points of the original curve at P, will meet that curve in $n-k+2$ other points, and therefore will meet also the locus of R in $n-k+2$ points. This locus is, therefore, of order $n-k+2$, as was to be proved.

The fixed points through which the variable $(k-2)$ way locus passes need not all be united at P, but they may be any $k-2$ arbitrary points on the curve.

We will now consider some examples of this remark.

1. *Unicursal curve of order* n *in* n*-dimensional space.*

A curve of order n *in flat space of* n *dimensions (and no less) is always unicursal.*—We may prove this independently by considering a variable $(n-1)$ flat which passes through $n-1$ fixed points on the curve. Its equation will be of the form $A+tA'=0$, where t is a variable parameter, and it will meet the curve in one other point, which is thus associated with a value of t.

The equations to such a curve may always be written in the form—

$$0=\left\| \begin{matrix} A, & B, & C \ldots K \\ B, & C, & D \ldots L \end{matrix} \right\| \text{.....................(1)},$$

where the $A, B, C \ldots K, L$ are linear functions of the coordinates, and the number of columns is $=n$. For the $n+1$ homogeneous coordinates are proportional to rational integral functions of t of the n^{th} order. Solving these $n+1$ equations for $1, t, t^2 \ldots t^n$ we find

$$1, t, t^2 \ldots t^n = A, B, C \ldots L,$$

which is equivalent to the system written down above.

The more general system of equations—

$$0=\left\| \begin{matrix} A, & B \ldots K \\ A', & B' \ldots K' \end{matrix} \right\| \text{.....................(2)},$$

where the $A \ldots K$, $A' \ldots K'$ are linear functions as before, may always and easily be reduced to the former, for they are got by eliminating t from the n equations.

$$\begin{aligned} A+tA'&=0, \text{ ....(3).} \\ B+tB'&=0, \\ &\vdots \\ K+tK'&=0. \end{aligned}$$

We may, however, solve these equations for the ratios of the coordinates, which will thus be expressed as rational functions of t of the n^{th} order. Solving these for $1, t, t^2 \ldots t^n$ we come back to the previous system.

The equations (3) exhibit the curve as the locus of the intersection of corresponding elements in n projective pencils.

The equation to the $(n-1)$ flat which passes through the n points whose parameters are $t_1, t_2 \ldots t_n$, is easily seen to be—

$$0=\left| \begin{matrix} A, & B, & C, & \ldots L \\ 1, & t_1, & t_1^2, & \ldots t_1^n \\ 1, & t_2, & t_2^2, & \ldots t_2^n \\ \vdots & & & \\ 1, & t_n, & t_n^2, & \ldots t_n^n \end{matrix} \right|.$$

But this equation is manifestly divisible by the coefficient of L, which is the product of the differences of all the t. If we write—

$$\begin{aligned} \Sigma_1 &= t_1+t_2+t_3+\ldots+t_n, \\ \Sigma_2 &= t_1t_2+t_1t_3+t_2t_3+\ldots+t_{n-1}t_n, \\ \text{etc.} &= \text{etc.} \\ \Sigma_n &= t_1t_2 \ldots t_n, \end{aligned}$$

then the equation is

$$0=L-K\Sigma_1+\ldots \pm B\Sigma_{n-1} \mp A\Sigma_n \text{...........(4)}.$$

If we omit the suffixes of the t in this formula we obtain the equation to the osculant $(n-1)$ flat at the point t. Namely (beginning at the other end), it is—

$$0 = At^n - nBt^{n-1} + \tfrac{1}{2}n(n-1)\,Ct^{n-2} - \ldots \pm nKt \mp L \ldots\ldots (5),$$

and we see at once that *the class of such a curve is always equal to its order.*

We thus obtain a very useful representation (*Abbildung*) of the points of the n-dimensional space by means of groups of n points on such a unicursal curve, namely, each point in the space is represented by the points of contact of the n osculant $(n-1)$ flats which pass through it. The use of such a representation of ordinary three-dimensional space by means of a skew cubic was pointed out by Dr Hirst, and the corresponding representation of a plane by means of a conic has been used by M. Darboux ('Sur une classe remarquable de courbes et de surfaces algébriques,' Paris, 1873, Note II., p. 183), and by me ("On the Transformation of Elliptic Functions," *Proc. Lond. Math. Soc.*, Vol. VII. (1875) [xxii., xxiii., pp. 205—228, supra]). It may be worth while to mention that an extension to all space of the theory of the in-and-circumscribed polygon may be obtained by this means.

A curve of this kind determines also a dualistic correspondence in the space of n dimensions. Through every point may be drawn n osculant $(n-1)$ flats, and through their points of contact another $(n-1)$ flat, which shall be called the *polar* of the point. If the point moves along a straight line its polar will pass through a fixed $(n-2)$ flat, the *polar* of the line. And generally if the point lies in any k flat the polar will pass through a fixed $(n-k-1)$ flat.

When $n=2$ we have the ordinary system of polar reciprocals in regard to a plane conic. When $n=3$ we have that system in regard to a skew cubic which is described by Schröter, *Crelle*, Vol. LXV. p. 39. These two systems are typical respectively of the cases in which n is even and odd. When n is even, the relation between the coordinates of two points, which expresses that each lies in the polar of the other, is a symmetrical one; consequently those points which lie in their own polars are points on a certain quadric locus, and the system is merely

that of the poles and polars in regard to this quadric locus upon which the curve lies. The equation to this locus is at once obtained by equating to zero the quadrinvariant of the form $(1, t)^n$ which occurs in the equation (5) of the osculant $(n-1)$ flat, namely, it is

$$0 = AL - nBK + \tfrac{1}{2}n(n-1)\,CH - \text{etc.} \ldots\ldots\ldots\ldots (6).$$

To prove this, observe that if in the equation (5) we substitute the coordinates of any point p, the values of t which satisfy the equation are the parameters of the points of contact of the osculant $(n-1)$ flats which pass through the point. If $t_1, t_2, \ldots t_n$ be these values, the equation (4) represents the $(n-1)$ flat which passes through the points of contact, that is to say, the polar of the point. Now if we denote by $A', B', \ldots$ the results of substituting the coordinates of the point p in $A, B, \ldots$ then we shall have—

$$A'\Sigma_1 = nB' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$
$$A'\Sigma_2 = \tfrac{1}{2}n(n-1)\,C',$$
$$\vdots$$
$$A'\Sigma_n = L'$$

so that, when n is even the equation of the polar is—

$$0 = AL' + A'L - n(BK' + B'K) + \tfrac{1}{2}n(n-1)(CH' + C'H) - \text{etc.} \ldots (8),$$

that is, it is simply the polar of the point in regard to the quadric (6).

It is to be observed that the quadric is completely determined when the curve is given. I reserve the question of the conditions to which the curve is subject when the quadric locus is given, or, say, the discussion of the problem to represent the relation of poles and polars in regard to a quadric locus (in space of an even number of dimensions) by means of a unicursal curve.

But when n is odd, the last term of equation (4) is negative, and the equation of the polar is—

$$0 = AL' - A'L - n(BK' - B'K) + \tfrac{1}{2}n(n-1)(CH' - C'H) - \text{etc.} \ldots (9),$$

that is, it is skew symmetrical, and *every point lies upon its polar.* It is convenient to use the term *co-flat* for $n+1$ points, which are in the same $(n-1)$ flat; with this nomenclature we may say that *when* n *is odd every point is co-flat with the* n *points of contact of the osculant* $(n-1)$ *flats, which can be drawn through it.* This will be recognised as an extension of the property of a skew cubic, that every point in space is co-planar with the points of contact of the three osculating planes which can be drawn through it.

A case of this skew symmetrical relation is given by any arbitrary state of motion of the whole space as a rigid body, the relation between two points being that the line joining them moves perpendicularly to itself. The polar of any point is an $(n-1)$ flat drawn through it perpendicular to the direction of its motion. When n is even there is always one point which remains at rest, and all the polars pass through this point. Thus the general motion of a solid in an even number of dimensions always depends in this simple way on the motion in one dimension less. In an odd number of dimensions, however, every point moves in the general case; but if any point is at rest, then all the points in a certain straight line are at rest.

Besides its order and class, a curve has, in general, characteristic numbers intermediate to these, which may be called its first rank, second rank, etc. The first rank is the order of the locus traced out by straight lines through two consecutive points of the curve; the second rank, of that traced out by planes through three consecutive points; and generally the k^{th} rank is the order of the $(k+1)$ wide locus traced out by k-flats through $k+1$ consecutive points. For the curve just considered these numbers are $2(n-1)$, $3(n-2)$, ... $(k+1)(n-k)$; it is convenient to derive them from the corresponding numbers for its projection, the unicursal curve of order n in $n-1$ dimensions, to which we now proceed.

2. *Unicursal curve of order* n *in* n − 1 *dimensions.*

Every curve of order n *in flat space of* n − 1 *dimensions is either unicursal or elliptic.* For it may be represented point-for-point on a plane cubic.

We shall treat these two cases in succession. They are exemplified by the two species of quartics in ordinary tri-dimensional space.

The coordinates of a point on the unicursal curve are proportional to rational integral functions of a parameter t. This representation may be simplified in a manner due to Rosanes, *Crelle*, Vol. LXXV. p. 166. We have n binary quantics of order n; now these may be linearly combined in n different ways so as to produce a perfect n^{th} power. Hence the original quantics may be expressed each as a linear function of the n^{th} powers of the same n linear quantics. Thus, for example, three binary cubics may be simultaneously reduced to the forms

$$au^3+b\,v^3+c\,w^3,$$
$$a'u^3+b'\,v^3+c'\,w^3,$$
$$a''u^3+b''v^3+c''w^3,$$

where $u+v+w=0$ identically. If the x, y, z of a point in a plane are respectively proportional to these cubics, we may, by solving the equations for u^3, v^3, w^3, obtain three linear functions X, Y, Z of the coordinates, which are respectively proportional to u^3, v^3, w^3. Transforming them to the new triangle whose sides are $X=0$, $Y=0$, $Z=0$ we must have the equation of a unicursal cubic expressed in the form

$$X^{\frac{1}{3}}+Y^{\frac{1}{3}}+Z^{\frac{1}{3}}=0.$$

It is clear that the lines $X=0$, $Y=0$, $Z=0$ are tangents at the three points of inflexion.

In general, let the n quantics be

$$a_0+na_1t+\ldots+a_nt^n,$$
$$b_0+nb_1t+\ldots+b_nt^n,$$
$$\vdots$$
$$h_0+nh_1t+\ldots+h_nt^n,$$

then the linear quantics $u, v, w \ldots$ are the factors of

$$\begin{vmatrix} t^n, & -nt^{n-1}, & \tfrac{1}{2}n(n-1)\,t^{n-2}, & \ldots\, t \\ a_0, & a_1, & a_2, & \ldots a_n \\ b_0, & b_1, & b_2, & \ldots b_n \\ \vdots & \vdots & \vdots & \vdots \\ h_0, & h_1, & h_2, & \ldots h_n \end{vmatrix}.$$

Since there are $n-2$ identical relations between n linear quantics, the $n-2$ equations of the unicursal curve may be written in the form

$$\left\| \begin{matrix} X_1^{\frac{1}{n}}, & X_2^{\frac{1}{n}}, & \ldots X_n^{\frac{1}{n}} \\ \alpha_1, & \alpha_2, & \ldots \alpha_n \\ \beta_1, & \beta_2, & \ldots \beta_n \end{matrix} \right\| = 0;$$

it is evident that the equations $X_1, X_2, \ldots X_n = 0$ represent stationary osculant $(n-2)$ flats, that is to say, $(n-2)$ flats which pass through n consecutive points of the curve.

The properties of this curve may be very conveniently studied by regarding it as a projection of the curve considered in the last section. If all the points of that curve be joined to a point O, not situated upon it, the joining lines will form a cone of order n; and on cutting this cone by an $(n-1)$ flat we shall obtain the curve now under discussion.

The n points of superosculation, whose existence has just been proved, are then clearly the projections of the points of contact of osculant $(n-1)$ flats to the full-skew curve drawn through the point O. It follows that *when* n *is odd, these* n *points of superosculation are on the same* (n − 2) *flat;* but when n is even this is not the case, unless the point O lies on the quadric locus associated with the full-skew curve, in which case we have a special variety of the projection. Thus the three points of inflexion of a nodal cubic are in one straight line; but a unicursal skew quartic in ordinary space has not in general the property that the points of contact of its four stationary osculating planes are in one plane. The property established above for the full-skew curve shews that this will be the case if the four points form an equianharmonic system, or if the quadrinvariant of the quartic which determines them is equal to zero. And generally when n is even, the n points of superosculation will be co-flat if, and only if, the quantic in t which determines them has its quadrinvariant zero.

By using the values of the coordinates of a variable point of the curve expressed in terms of a parameter t, we may obtain an expression of this quadrinvariant and also of its product by

the discriminant in terms of the roots of the quantic. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the values of t which belong to the points of superosculation, and $x_1, x_2, \ldots x_n$ the coordinates of a variable point on the curve. Then we may write

$$x_i = (t-\alpha_i)^n, \qquad i = 1, 2, \ldots n,$$

and the coordinates of the point α_i are $(\alpha_i-\alpha_1)^n$, $(\alpha_i-\alpha_2)^n$, $\ldots (\alpha_i-\alpha_n)^n$. If for shortness we write (hk) instead of $\alpha_h - \alpha_k$, then the condition that the n points shall be co-flat is

$$0 = \begin{vmatrix} 0\ , & (12)^n, & (13)^n, & \ldots (1n)^n \\ (21)^n, & 0\ , & (23)^n, & \ldots (2n)^n \\ (31)^n, & (32)^n, & 0\ , & \ldots (3n)^n \\ \vdots & \vdots & \vdots & \vdots \\ (n1)^n, & (n2)^n, & (n3)^n, & \ldots\ 0 \end{vmatrix}.$$

This is obviously always satisfied if n is odd, for then the determinant is skew symmetrical, and being of odd order it necessarily vanishes. If, however, n is even, the determinant is a symmetrical function of the roots which vanishes when any two of them are equal; and consequently it must contain as a factor the product of the squares of their differences. Now the determinant is of the order $2n$ in each root, and the discriminant is of order $2(n-1)$; therefore the remaining factor is of order 2 in each root, and being a symmetrical invariant must be a function of the squares of their differences. It can therefore be no other than $\Sigma (\alpha_1-\alpha_2)^2 (\alpha_3-\alpha_4)^2 \ldots (\alpha_{n-1}-\alpha_n)^2$; this is, to a factor *près*, equal to the quadrinvariant of the form

$$(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_n).$$

The equation to the $(n-2)$ flat passing through two consecutive points of the curve at t, and through $n-3$ other points $p\,q \ldots u$, is clearly

$$0 = \begin{vmatrix} x\, dx\, p\, q \ldots y \\ 1\ 2\ 3\ 4 \ldots n \end{vmatrix},$$

where the y are current coordinates, and the determinant is expressed in umbral notation. Writing in this for x_i, $(t-\alpha_i)^n$, and for dx_i, $n(t-\alpha_i)^{n-1}dt$, we may observe that the determinant

$$\begin{vmatrix} (t-\alpha_1)^n\ , & (t-\alpha_2)^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1} \end{vmatrix} = (\alpha_2-\alpha_1)(t-\alpha_1)^{n-1}(t-\alpha_2)^{n-1},$$

so that the equation is of order $2(n-1)$ in t. It thence follows that $2(n-1)$ different $(n-2)$ flats may be drawn through $n-2$ arbitrary points to touch the curve; or that the developable traced out by the tangent lines is of the order $2(n-1)$.

Similarly, from the value of the determinant

$$\begin{vmatrix} (t-\alpha_1)^n, & (t-\alpha_2)^n, & \dots (t-\alpha_{k+1})^n \\ (t-\alpha_1)^{n-1}, & (t-\alpha_2)^{n-1}, & \dots (t-\alpha_{k+1})^{n-1} \\ \vdots & \vdots & \vdots \\ (t-\alpha_1)^{n-k}, & (t-\alpha_2)^{n-k}, & \dots (t-\alpha_{k+1})^{n-k} \end{vmatrix},$$

which is equal to the product of the differences of $\alpha_1, \alpha_2, \dots \alpha_{k+1}$ multiplied by

$$\{(t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_{k+1})\}^{n-k},$$

we may conclude that the number of $(n-2)$ flats which can be drawn through k consecutive points of the curve and through $n-k$ other arbitrary points is $(k+1)(n-k)$; or that the k-wide locus which is traced out by $(k-1)$ flats passing through k consecutive points is of the order $(k+1)(n-k)$. For the equation of an $(n-2)$ flat passing through k consecutive points is clearly

$$0 = \begin{vmatrix} x\, dx\, d^2x \dots d^{k-1}x\, p\, q \dots y \\ 1\ 2\ 3 \dots k, \qquad \dots n \end{vmatrix},$$

where we must substitute for the x_i, dx_i, d^2x_i, etc., the descending powers of $t-\alpha_i$ beginning at the n^{th}. Making k equal to $n-1$ we obtain the equation of the osculant $(n-2)$ flat at any point of the curve; it is

$$0 = \frac{P_1 y_1}{(t-\alpha_1)^2} + \frac{P_2 y_2}{(t-\alpha_2)^2} + \dots + \frac{P_n y_n}{(t-\alpha_n)^2},$$

where P_i = product of the differences of all the α except α_i. Thus the class of the curve is $2(n-1)$.

3. *Unicursal curve of order* n *in* n − k *dimensions.*

The characteristic numbers belonging to this curve may at once be obtained by regarding it as a projection of the full-skew curve. The number of ranks is $n-k-2$, and the numerical values of them are respectively $2(n-1)$, $3(n-2)$, ... $k(n-k+1)$; the class is $(k+1)(n-k)$; and the number of points of superosculation is $(k+2)(n-k-1)$. For example, the unicursal

quintic in three dimensions is of rank $2.4, = 8$, and of class $3.3, = 9$, and it has $4.2, = 8$ superosculant planes.

Convenient forms of the equations may be got by eliminating some of the variables from the equations of the full-skew curve; but care must be taken to select these variables so that the resulting system is sufficiently general.

4. *Elliptic (or bicursal) curve of order* n *in* n − 1 *dimensions.*

We have proved already that a curve of order n in $n-1$ dimensions can be represented, point for point, on a plane cubic. If, therefore, it is not unicursal, its coordinates can be expressed in terms of elliptic functions of a single parameter. Now, it follows from the investigations of Clebsch, *Crelle*, Vol. LXIV. (1864), pp. 210—270, that if n points of the curve are co-flat, the sum of their parameters will differ from a certain constant by a sum of integer multiples of the two periods of the elliptic function. Let the periods be ω and ω', then if $t_1, t_2, \dots t_n$ are the parameters of the points,

$$t_1 + t_2 + \dots + t_n = c + a\omega + b\omega',$$

where c is a constant, and a, b are integers. To find the points of superosculation, we must suppose the n points to become identical, or the t, still satisfying this equation, to become equal. We thus obtain

$$nt = c + a\omega + b\omega'$$

$$t = \frac{c}{n} + \frac{a}{n}\omega + \frac{b}{n}\omega',$$

and values of t, representing distinct points, will be got by giving to the numbers a, b the values $0, 1, \dots n-1$ independently. Hence *there are* n^2 *points of superosculation.*

Thus a plane cubic has nine inflexional tangents, and a quadri-quadric curve has sixteen superosculant planes.

Propositions hold good in general in regard to the grouping of these points, which are analogous to those which relate to the inflexions of a cubic. Thus, *an* (n − 2) *flat drawn through* n − 1 *of them will always pass either through another besides, or through the tangent line at one of the* n − 1. This is obvious from the values already given for the parameters of points of superosculation.

Through any point of the curve can be drawn $(n-1)^2$ osculant $(n-2)$ flats. This is proved in the same way as the preceding proposition, which is, in fact, the projection of it; for if through the given point we draw a cone containing the curve, and cut it by an $(n-2)$ flat, the section will be an elliptic curve of order $n-1$ in $n-2$ dimensions, and the projections of the points whose osculant $(n-2)$ flats pass through the given point will be points of superosculation on the projected curve. Hence, also, the lines joining the given point to the points of contact are grouped in respect of co-flatness in the same way as the points of superosculation in the curve of next lower order.

More generally, through k given points of the curve there can be drawn $(n-k)^2$ $(n-2)$ flats which have $(n-k)$ pointic contact with the curve. If $u_1, u_2, \ldots u_k$ are the parameters of the k given points, those of the required points are given by

$$u = \frac{1}{n-k}(u_1 + u_2 + \ldots u_k + a\omega + b\omega'),$$

where the integers a, b may take independently the values

$$0, 1, \ldots n-k-1.$$

From these results we may now determine the various ranks and the class of the curve. Suppose that we know the number of $(n-2)$ flats which can be drawn through $n-2$ arbitrary points in space—or, which is the same thing, through an arbitrary $(n-3)$ flat P— to touch a certain curve. Then, if the arbitrary $(n-3)$ flat meets the curve in any point, *two* of these will coincide at that point. For taking an $(n-4)$ flat in the $(n-3)$ flat, and joining it to all the points of the curve by $(n-3)$ flats, we may cut this figure by a plane or 2 flat. Every $(n-3)$ flat will cut this plane in a single point. The problem is then reduced to drawing tangents from a point (viz., the intersection of P by the plane) to a plane curve; and we know that when this point lies on the curve, two of the tangents coincide at it.

In general, a certain number of $(n-2)$ flats can be drawn

through an arbitrary $(n-k-1)$ flat to have k-pointic contact with a given curve; this number is, in fact, the $(k-1)^{\text{th}}$ rank of the curve. If the arbitrary $(n-k-1)$ flat meets the curve at any point, then k of these $(n-2)$ flats coincide at that point. For we may project the whole figure from an $(n-k-2)$ flat lying in the $(n-k-1)$ flat on to a k flat. The problem is then reduced to drawing $(k-1)$ flats through a given point to have k-pointic contact with a curve in k dimensions. Now we know, from the example of the full-skew curve, that, when the point lies on the curve, k of these coincide at the point.

If the arbitrary $(n-k-1)$ flat meet the curve in more points than one, k of the osculants will coincide at each of them; and this result is not affected by the union of the points into one. In particular, if it meet the curve in $n-k$ coincident points, the number of osculants which there coincide is $k(n-k)$.

Applying now these general considerations to the elliptic curve, we find at once that the $(k-1)^{\text{th}}$ rank of it is nk. For we must add to the number $k(n-k)$, just obtained, the number, k^2, given by the theory of elliptic functions for the $(n-2)$ flats drawn through $n-k$ consecutive points of the curve to have k-pointic contact elsewhere. In particular, the class of the curve is $n(n-1)$; we have observed already that the number of superosculants is n^2.

Thus, a plane cubic is of order 3, class 6, and has 9 inflexions; a quadri-quadric is of order 4, rank 8, class 12, and has 16 superosculant planes. We learn, moreover, that a quintic curve in four dimensions, when not unicursal, is of first rank 10, second rank 15, class 20, with 25 points of superosculation. Hence a quintic in three dimensions, with five apparent dps., is of rank 10, class 15, and has 20 superosculant planes; this follows by projection from the former case.

A curve of this kind, viz., an elliptic curve of order n in an $(n-1)$ flat has its coordinates $x_1, x_2 \ldots x_n$ determined by the equations

$$x_1, x_2, \ldots x_n = 1, t, t', t^2, tt' \ldots$$

(the last term on the right being $t^{\frac{1}{2}(n-1)}t'$ or else $t^{\frac{1}{2}n}$, according as n is odd or even), where $t = \operatorname{sn}^2(u + iK')$ and

$$t' = \frac{dt}{du} = 2 \operatorname{sn}(u + iK') \operatorname{cn}(u + iK') \operatorname{dn}(u + iK')$$
$$= \sqrt{2t(1-t)(1-k^2t)}.$$

{If n is even, we may write $t = \operatorname{sn}^2 u$ instead of $\operatorname{sn}^2(u + iK')$.} The condition for n points $u_1, u_2 \dots u_n$ to be co-flat is then

$$u_1 + u_2 + \dots + u_n = 0.$$

See Lindemann; Clebsch's *Lectures on Geometry*, vol. II.

{*Theory of Derived Points on an Elliptic (or Bicursal) Curve.*

Sylvester's theory of derived points on a plane cubic is as follows:—Starting from any given point on the curve, we may construct its *tangential*, or point where the tangent at the original point meets the curve again; similarly we may construct the tangential of the tangential, or second tangential, and so on. By joining any two non-consecutive points on this series, we can find their *residual*, the point where the joining line meets the curve again. In this way we obtain an infinite group of points derived from (and including) the original point, such that the line joining any two of them is either tangent at one of these or passes through a third point of the group. It is to be observed that all points on the curve uniquely derived from the given point by any geometrical process (*e.g.*, the point where the conic of five-pointic contact meets the curve again, the point where cubics of eight-pointic contact meet the curve again, &c.) are included in the group.

The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, and the order of the functions to which they are proportional is always a square number. Thus the three coordinates of the tangential are proportional to quartic functions of the coordinates of the original. If the square root of the order of these functions be called the order of the derived point, then we have the theorem that when three derived points are in a straight line, the order of one of them is equal to the sum of the orders of the other two. It is observed that there is no derived point whose order is divisible by 3. By help of this observation it is easy to make out a scheme of the orders; for when we join two points, the order of their residual must be the sum or the difference of the orders of the points, and one or the other of these is always divisible by 3.

This theory is really a geometric representation of the multiplication of elliptic functions. The coordinates x_1, x_2, x_3 of any point on the cubic curve may by proper choice of axes be made proportional to elliptic functions of a parameter u, so that $x_1 : x_2 : x_3 = 1 : \operatorname{sn}^2(u + iK') : \operatorname{sn}(u + iK') \operatorname{cn}(u + iK') \operatorname{dn}(u + iK')$. This being so, if u, v, w are parameters of three points in a straight line, we shall have $u + v + w = 0$. If v be the tangential of u, the three points u, u, v are in a straight line, and $2u + v = 0$, or $v = -2u$. Hence the series of tangentials has for parameters

$$u, \ -2u, \ +4u, \ -8u, \ \&c.:$$

and in general the parameter of any derived point is of the form nu, where n is a positive or negative integer. The number n, taken positively, coincides with what was called the order of the derived point. For the elliptic functions of nu are of the order n^2 in the elliptic functions of u.

In this way all points of the theory are explained, excepting the fact that no derived point has its order divisible by 3.

Moreover, we see at once that the theory can be extended to other curves of deficiency 1; as, for example, the quadri-quadric curve. Starting with any point on this curve, we may find the point where the osculating plane at that point meets the curve again; then repeat the process with the point so found, and so on. The plane joining any three of these points will meet the curve in another derived point, or else touch it at one of the three points. The plane drawn through one derived point to touch the curve at another derived point will meet it again in a derived point, *or* touch at the first point, *or* osculate at the second. The coordinates of any derived point are of the order n^2 in those of the original point, where $\pm n$ may be called the order of the derived point. In this case the order of no derived point is divisible by 2.

I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* (in the sense explained in the beginning of this paper) of deficiency greater than 1. For this purpose I considered scrolls. Taking an arbitrary generator on a quartic scroll having two linear directrices, we may draw a one-sheeted hyperboloid through three consecutive generators at that place; this will meet the quartic scroll in one other generator, which is thus uniquely derived from the given one. Similarly on a quintic scroll contained in a linear complex, the two tractors of four consecutive generators meet the scroll in two other points lying on a generator. And on a sextic scroll not contained in a linear complex, the linear complex having five-line contact at a given generator (containing five consecutive generators) will contain one other generator of the scroll. In these three cases, then, from any three, four, or five generators we may uniquely derive a fourth, fifth, or sixth generator respectively; and the whole theory of derived elements may be applied to the generators of these scrolls.

Unfortunately, however, each of the scrolls considered is at most of deficiency 1, so that we merely get more illustrations of the multiplication of elliptic functions. And it may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1.

Suppose that it has no singular points, and that $k-1$ points on it being given, there is uniquely determined one other point. If this is effected (as in the above examples) by drawing a flat space through the $k-1$ points, which meets the curve in one other point, then it must be of the order k. Moreover, it must be in a flat space of so many dimensions that the flat of one dimension less is determined by $k-1$ points. Now a $(k-2)$ flat is determined by $k-1$ points; therefore, the curve is in a $(k-1)$ flat.

Thus the impossibility of extending the theory of derivation

to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k-1$ dimensions is at most of deficiency 1. This failure was the starting point of the present paper.

It remains to explain why, in the group of numbers expressing the orders of the derived points, only certain forms present themselves. Let that number which, with $k-1$ other numbers, makes up zero, be called the *residual* of those numbers; it is, in fact, their sum taken negatively. Then the process of forming the group is to start from unity, and add the residual of every $k-1$ numbers of the group, repetitions being allowed. I say that by this process we shall only get numbers of the form $mk+1$. For let m_1k+1, m_2k+1, &c. be $k-1$ such numbers, then their residual is $-(m_1+m_2+..)\,k-k-1$, which is a number of the same form. Now as unity, with which we start, is of this form, it follows that all the numbers of the group must be of the form $mk+1$.—January, 1879.}

CURVES OF DEFICIENCY p.

Theorems relating to Abelian Functions.

It will be convenient to put together shortly those propositions relating to the application of Abelian functions to curves which will be wanted in the sequel.

The aggregate of the real and imaginary points on a curve constitutes a two-way spread, or surface, which may be transformed, by stretching without tearing, into the surface of a body with p holes in it. On this surface there are $2p$ distinct closed curves which cannot without breaking be shrunk into a point, namely, one round each hole, and one through each hole. Any other irreducible circuit must be made up of combinations of these.

If any rational function of the coordinates be integrated from one point to another along the curve-spread, the value of the integral will depend upon the path of the integration. If the integral becomes infinite at any points, it may be altered in value by making the path go round one or more of these; but in any case it may be altered by incorporating into the path any of the $2p$ closed circuits just mentioned. It is found that there are p distinct rational functions of the coordinates whose integrals do not become infinite at any point of the curve-spread. Any other integral which is everywhere finite must be a linear combination of these. Of such linear combinations it is convenient to take a certain set as the *normal* set; they are so chosen that each of them, when integrated along a closed path which goes *round* a hole, gives zero for all the holes but one, and πi for that one; thus, the p integrals, which we may call $u_1, u_2, u_3, \ldots u_p$, are associated one by one with the p holes $1, 2, \ldots p$. If they are integrated along a closed curve passing *through* the hole h, let the values be called a_{h1}, a_{h2}, a_{hp}; then it is found that $a_{hk} = a_{kh}$, or the integral of u_h through the hole k is equal to the integral of u_k through the hole h.

If we now take all the integrals from a point x to a point y along the same path, and if $u_1, u_2, \ldots u_p$ are the set of values for one such path, and $U_1, U_2, \ldots U_p$ for another path, then we must have

$$\begin{aligned} U_1 &= u_1 + m_1\pi i + q_1 a_{11} + q_2 a_{12} + \ldots + q_p a_{1p}, \\ U_2 &= u_2 + m_2\pi i + q_2 a_2 + q_2 a_{22} + \ldots + q_p a_{2p}, \\ &\vdots \\ U_p &= u_p + m_p\pi i + q_1 a_{p1} + q_2 a_{p2} + \ldots + q_p a_{pp}, \end{aligned}$$

where the numbers m, q are integers; namely, m_h is the additional number of times the new path has gone round the hole h, and q_h is the additional number of times it has gone through that hole. We shall write these equations thus

$$U_1, U_2, \ldots U_p \equiv u_1, u_2 \ldots u_p \pmod{\pi i, a},$$

and shall say that the quantities U are *congruent* to the quantities u in respect of the periods πi, a.

Suppose now that the curve has no actual nodes, and that a locus of any order intersects it in the points $x_1, x_2, \ldots x_m$. Then, if another locus of the same order intersects it in the points $y_1, y_2, \ldots y_m$, and we take any one of the integrals, say u,

from x_1 to y_1, from x_2 to $y_2, \ldots$ from x_m to y_m, the sum of these results will be congruent to zero. That is to say

$$\Sigma \int_x^y du_h \equiv 0.$$

Here the Σ refers to the suffixes of the x and y, not to h. There are p such equations. This is Abel's Theorem.

When the curve is in a k-flat and of the order n, we shall use this theorem chiefly for its n intersections with a $(k-1)$ flat. If we regard the lower limits of the integrals (the point x) as fixed, the integrals for any point y may be regarded as parameters belonging to that point, and then Abel's Theorem gives us p equations between the parameters of n points which lie on a $(k-1)$ flat. The truth of these equations is *necessary* to the points lying on a $(k-1)$ flat, but it may not be sufficient. Thus in a bicircular quartic curve, $p=1$, we have one equation to express that four points are in a straight line, and if the points are collinear the equation is true. But it does not follow from the equation that the points are collinear; in fact, the equation holds equally good if the points are in a circle.

If the sums of the parameters of p points are given, that is, if we have the p equations—

$$\left(\int^{x_1} + \int^{x_2} + \ldots + \int^{x_p}\right) du_h = v_h \quad \{h = 1, 2, \ldots p\},$$

the v_h being arbitrary constants, and the lower limits of the integrals being supposed constant; then the upper limits $x_1, x_2, \ldots x_p$ may be expressed in terms of the quantities v_h —namely, they are the roots of an equation of degree p whose coefficients are products of ϑ-functions of the v. If

$$\phi(m_1, m_2, \ldots m_p) = \Sigma m_h m_k a_{hk} + 2\Sigma m_h v_h,$$

then

$$\vartheta(v_1, v_2, \ldots v_p) = \Sigma^p e^{\phi(m)},$$

the Σ^p indicating that each of the p integers $m_1, m_2, \ldots m_p$ is to take all integral values positive and negative. When the lower limits are so chosen that the sum of the parameters is zero for the complete intersection by any locus, this ϑ-function has remarkable properties. If we sum each of the parameters for any $p-1$ points on the curve, the ϑ-function whose arguments are these sums will vanish. That is if

$$v_h = \left(\int^{x_1} + \int^{x_2} + \dots \int^{x_{p-1}}\right) du_h,$$

then $\vartheta(v_1 v_2 \dots v_p) = 0$. If these sums are taken for any $p-2$ points, not only will the ϑ vanish, but also its differential coefficient in regard to any one of the points. And generally, if we take for the v the sums of the parameters for $p-r$ points, the ϑ and its first $r-1$ differential coefficients in regard to any of the points will vanish.

Conversely, if the p quantities v are such that $\vartheta(v)$ and its first $r-1$ differential coefficients vanish, then it is possible to find $p-r$ points x such that

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{p-r}}\right) du_h = v_h.$$

Although here the number of equations is greater by r than the number of unknown quantities, yet it is possible to satisfy them all in virtue of the relations existing between them.

Relation between the Order and Deficiency of a Curve.

We shall now apply these theorems to the study of curves existing in k dimensions, of the order n and deficiency p. A $(k-1)$ flat cuts such a curve in n points, such that the sum of each of the p parameters, for the n points, is zero. But a $(k-1)$ flat is determined by k points; so that, k arbitrary points being chosen on the curve, it is always possible to find $n-k$ other points, so that the sum of each parameter for the whole n points shall be zero. Let then $-v_1, -v_2, \dots -v_p$ be the sums of the parameters for the given k points; then to find the remaining $n-k$ points we have the p equations

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{n-k}}\right) du_h \equiv v_h.$$

If p is not greater than $n-k$, we know that these equations can be solved, although the solution may be indeterminate. But if $p > n-k$, the equations cannot be solved unless certain

conditions are satisfied by the v. Let $p-n+k=r$, then r conditions must be satisfied; namely the v must be sums of the parameters of not more than $p-r$ (or $n-k$) points. But they are sums of the parameters of k points; therefore k is not greater than $n-k$, or $2k$ is not greater than n. We have proved then that

If p > n − k, *then* 2k *is not greater than* n.

Conversely, if k > ½n, p *is at most equal to* n − k.

We may also state the proposition in this way. *A curve of order* n *and deficiency* p, *not greater than* ½n, *can at most exist in* n − p *dimensions.*

{It appears, therefore, that the theorems at the beginning of the paper may be extended, and that in n dimensions we have the curve of order n which is unicursal, the curve of order $n+1$, and deficiency at most 1, of order $n+2$, and deficiency at most 2, and so on till we come to the order $2n$, which is the first case of exception, and may have deficiency $n+1$. This curve is the natural geometric representation of the general Abelian functions, its multiple tangent flats playing the same part as the double tangents of the quartic curve in Riemann's beautiful paper on the case $p=3$. H. Weber has noticed that in four dimensions this curve is the complete intersection of three quadric loci.—January, 1879.}

(ABSTRACT OF XXXIII)*.

"A CURVE," is to be understood to mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and if these be connected by $k-1$ equations

* [From the *Proceedings of the Royal Society*, No. 187, 1878. This abstract, I believe, contains the communication on the same subject made to the London Mathematical Society, Feb. 8th, 1877.]

of any order, the curve is either the aggregate of common solutions, or, when this breaks up into algebraically distinct parts, the curve is one of these parts.

In the paper, of which this is an abstract, theorems are established relating to the nature of the space in which such curves can exist, to the mode of representing them in flat space of lower dimensions, and to some of their properties. The following are the leading theorems:—

I. Every proper curve of the nth order is in a flat space of n dimensions or less.

II. A curve of order n in flat space of k dimensions (or less) may be represented, point for point, on a curve of order $n-k+2$ in a plane.

III. A curve of order n, in flat space of n dimensions (and no less), is always unicursal.

From this the author obtains a representation of the points of an n dimensional space by means of groups of n points on such a unicursal curve, corresponding to the methods of Hirst and Darboux for three-dimensional space.

When n is even, the system corresponds to that of poles and polars in regard to a quadric locus upon which the curve lies.

When n is odd, every point is co-flat (i.e. $n+1$ points lie in the same $n-1$ flat), with the n points of the osculant $(n-1)$ flats which can be drawn through it.

IV. Every curve of order n in flat space of $n-1$ dimensions is either unicursal or elliptic.

V. When the curve is unicursal, and n is odd, the n points of superosculation, or points of stationary osculant $(n-2)$ flats, are on the same $(n-2)$ flat. But when n is even, this will be the case only under a certain condition.

VI. When the curve is elliptic (or bi-cursal) the class of the curve is $n(n-1)$, and the number of superosculants n^2.

If we consider a curve of the order n and deficiency p, existing in k dimensions, a $(k-1)$ flat cuts such a curve in n points, such that the sum of each of the p parameters (Abel's theorem gives p equations between the parameters of n points which lie on a $(k-1)$ flat), for these n points is zero. And we obtain the theorem.

VII. A curve of order n and deficiency p, not greater than $\frac{1}{2}n$, can at most exist in $n-p$ dimensions.

*XXXIV.

ON THE POWERS OF SPHERES.*

DEF. The *power* of two spheres (or of one sphere in regard to the other) is the squared distance of their centres less the sum of the squares of their radii.

Or if d be the distance of the centres, r_1, r_2 the radii, it is

$$d^2 - r_1^2 - r_2^2,$$

which is the same thing as

$$-2r_1r_2 \cos\theta,$$

where θ is the angle of intersection.

If the power vanishes, the spheres cut at right angles.

Let the equations of two spheres be

$$P \equiv a(x^2+y^2+z^2) + 2bx + 2cy + 2dz + e = 0,$$
$$Q \equiv a'(x^2+y^2+z^2) + 2b'x + 2c'y + 2d'z + e' = 0,$$

then their power is equal to

$$\frac{ae' + a'e - 2bb' - 2cc' - 2dd'}{aa'} \quad \text{.................. (1).}$$

Let the numerator of this fraction be denoted by (PQ), and let O stand for the sphere $a=b=c=d=\theta$, $e=1$, (the sphere at infinity); then the fraction may be written

$$\frac{(PQ)}{(PO)(QO)} \quad \text{.......................... (2).}$$

* [I think this paper must be the one more than once promised me by Prof. Clifford, and that it contains what he communicated to the London Mathematical Society, Feb. 27, 1868, *Proc.* Vol. II. 61.]

Under the name *sphere* must be included as particular cases points and planes. The power of a point in regard to a sphere is the squared tangent to it; the power of two points is the squared distance between them. The powers of spheres in regard to planes are infinite quantities which are proportional to the distances of their centres from the planes. The powers of planes in regard to one another are infinities of the second order which are proportional to the cosines of the angles of intersection.

The symbol (PQ) is linear in regard to the coefficients of the two spheres involved, and therefore distributive. That is to say, we have

$$(P, \lambda Q + \mu R) = \lambda(PQ) + \mu(PR).$$

It follows directly from this that if we form a determinant whose constituents are such symbols, of the form

$$\binom{ABCDE}{PQRST} \equiv \begin{vmatrix} (AP), & (AQ), & (AR), & (AS), & (AT) \\ (BP), & \text{\&c.} & \dots & \dots & \dots \\ (CP), & & & & \\ (DP), & & & & \\ (EP), & & & & (ET) \end{vmatrix}$$

(the letters denoting any ten spheres) it must be equal to the product of the determinants $(ABCDE)$, $(PQRST)$, multiplied by a linear factor. The ordinary theorem for multiplication of determinants shews that we have in fact

$$\binom{ABCDE}{PQRST} \equiv 8\,(ABCDE)\,(PQRST) \quad \text{.......... (3).}$$

If we divide the rows of the determinant by (AO), (BO), etc. and the columns by (PO), (QO), etc., its constituents become actually the powers of the spheres involved. Now, for the right-hand side of the equation (3), we have

$$\frac{(ABCDE)}{(AO)(BO)(CO)(DO)(EO)}$$
$$= \frac{(ABCD.E)}{(ABCDO)(EO)} \cdot \frac{(ABCDO)}{(AO)(BO)(CO)(DO)}.$$

The quantity

$$\frac{(ABCDO)}{(AO)\,(BO)\,(CO)\,(DO)},$$

is easily seen to be 48 times the volume of the tetrahedron whose vertices are at the centres of the four spheres. The other factor

$$\frac{(ABCDE)}{(ABCDO)\,(EO)}$$

is clearly the power of the sphere E in regard to a sphere cutting A, B, C, D orthogonally. We see then that *given any five spheres, the product of the tetrahedron whose vertices are at the centres of any four into the power in regard to the fifth of a sphere cutting them orthogonally is a symmetrical function of the five spheres.* I shall call this the *apospheric function* of the five spheres; it vanishes when they are all orthotomic of the same sphere.

This being so, the theorem (3) informs us that *the determinant formed with the powers of two sets of five spheres is equal to* 6144 ($=8\times48\times48$) *times the product of their apospheric functions.*

The corresponding determinant formed with two sets of six spheres vanishes identically; or we have always

$$\binom{ABCDEF}{PQRSTU}\equiv 0 \;\ldots\ldots\ldots\ldots\ldots\; (4).$$

Power-Coordinates.

DEF. Coordinates of a sphere are quantities proportional to certain multiples of its powers in respect of five fixed spheres.

It is convenient for many purposes to take as these multiples the reciprocals of the radii of the fixed spheres. Thus if the coordinates of a sphere X are x_1, x_2, x_3, x_4, x_5, its powers in respect of the fundamental spheres are x_1r_1, x_2r_2, etc., where

r_1, r_2, etc., are their radii. Using the symbols 1, 2, 3, 4, 5 for the fundamental spheres, we have by (4)

$$\binom{1\;2\;3\;4\;5\;X}{1\;2\;3\;4\;5\;X}\equiv 0,$$

or, expanding and dividing rows and columns by the $r_1, r_2, \ldots$

$$\phi_r=\begin{vmatrix} 1, & \cos 12, & \cos 13, & \cos 14, & \cos 15, & x_1 \\ \cos 21, & 1, & \cos 23, & \&c. & & x_2 \\ \cos 31, & \cos 32, & 1, & & & x_3 \\ \cos 41, & \&c. & & 1, & & x_4 \\ \cos 51, & & & & 1, & x_5 \\ x_1, & x_2, & x_3, & x_4, & x_5, & -2r^2 \end{vmatrix}=0\ldots(5),$$

where r is the radius of the sphere X. This equation may be regarded as giving the radius of a sphere in terms of its coordinates. Let ϕ be the value of the determinant ϕ_r when $r=0$, or when the sphere reduces to a point. Then whenever the coordinates x represent a point, they must satisfy the homogeneous equation of the second order

$$\phi\,(xx)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6).$$

If we choose to regard the x as abbreviations for expressions of the form

$$\frac{(x-a)^2+(y-b)^2+(z-c)^2-r^2}{r},$$

then the equation (6) is identically satisfied; and from this point of view it is a form of the identical relation connecting the equations of any five spheres. But from our point of view the x are primarily coordinates of a sphere, and (6) is only satisfied when the sphere reduces to a point.

An expression for the power of two spheres X and Y is obtained in the same manner. Namely, the equation

$$\binom{1\;2\;3\;4\;5\;X}{1\;2\;3\;4\;5\;Y}=0,$$

gives at once

$$\text{power of } X \text{ in regard to } Y=\frac{\phi\,(xy)}{(1\;2\;3\;4\;5)^2}.$$

If this vanishes, X is orthotomic of Y. Hence any linear equation

$$\Sigma u_i x_i = 0 \quad (i = 1, 2, 3, 4, 5)$$

expresses that X is orthotomic of a fixed circle whose coordinates y are to be found by solving the equations

$$u_i = \frac{\partial \phi (yy)}{\partial y_i} = \frac{\partial \phi (xy)}{\partial x_i}.$$

If therefore we write

$$\Phi (u, u) = \Sigma u_i^2 + 2 \Sigma u_i u_j \cos (ij),$$

we shall have

$$(1\ 2\ 3\ 4\ 5)\ y_i = \frac{\partial \Phi (uu)}{\partial u_i}.$$

We may draw five new spheres, each orthotomic to four of the fundamental spheres; and it is clear that the quantities u form a system of coordinates relating to these reciprocal spheres. The system is not quite of the same kind as the original one; for the quantities u are proportional to the powers of the sphere U divided not by the radii of the reciprocal spheres, but by the volumes of the tetrahedra whose vertices are at the centres of the fundamental spheres. In one very important case, however, these ratios coincide, and the reciprocal spheres are the same as the fundamental spheres; namely, if these form an orthogonal system. In that case

$$\phi (xx) = \Sigma x_i^2 = \Phi (xx),$$
$$\phi (xy) = \Sigma x_i y_i = \Phi (xy),$$

and the coefficients in the equation of a sphere are also its coordinates.

*XXXV.

A FRAGMENT ON MATRICES.

[In explanation of the following paper observe that the matrix $\phi, = \begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix}$ substitutes for the point having the coordinates (α, β, γ), the point having the coordinates

$$(\phi \between \alpha, \beta, \gamma) = a\alpha + b\beta + c\gamma,\ a'\alpha + b'\beta + c'\gamma,\ a''\alpha + b''\beta + c''\gamma;$$

in particular for the points i, j, k coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, it substitutes the points whose coordinates are (a, a', a''), (b, b', b''), (c, c', c'') respectively. An indeterminate matrix (or more definitely, a matrix indeterminate in the first degree) is a matrix the determinant of which vanishes, but for which the first minors do not all of them vanish: such a matrix substitutes for a given point a point in a determinate line called in the paper, the axis: but for one position of the given point, called the null-point, the position of the substituted point is altogether arbitrary. A matrix indeterminate in the second degree is a matrix for which all the first minors vanish, or what is the same thing, one for which the second and third rows are mere multiples of the first row. C.]

An indeterminate matrix ϕ substitutes for the fundamental points i, j, k three collinear points $\alpha_1, \alpha_2, \alpha_3$. As these must satisfy an identical equation $l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3 = 0$, there is a point $l_1 i + l_2 j + l_3 k$ which is reduced by ϕ to zero. But forasmuch as upon the line α there is a (1, 1) correspondence (viz. to every point $x_1 i + x_2 j + x_3 k$ there corresponds one point $\Sigma x\alpha$, to every point $\Sigma x\alpha$ on the line corresponds the straight line

$$(x_1 + el_1)\, i + (x_2 + el_2)\, j + (x_3 + el_3)\, k$$

which has *one* point on the line α), there must be two united points, say μ, ν. Thus if λ represents the null-point and λ, μ, ν are now taken as fundamental points, the matrix ϕ is reduced to the form

$$\begin{vmatrix} 0, & 0, & 0 \\ 0, & g_2, & 0 \\ 0, & 0, & g_3 \end{vmatrix}.$$

This clearly corresponds to that mode of projection in which the centre of projection is taken on one of the two planes but not on the other.

Now consider two indeterminate matrices ϕ, ψ, and let λ, ρ be their null-points, μ, ν and σ, τ the respective united points. The product $\phi\psi$ will have ρ for a null-point, and $\mu\nu$ for axis; while $\psi\phi$ will have λ for null-point and $\sigma\tau$ for axis. In general therefore the product will still be indeterminate in the first degree only. But if λ is on the line $\sigma\tau$, i.e. if the null-point of ϕ is on the axis of ψ, then there is a certain line through ρ any point on which ψ will convert into λ; any point on which, therefore, $\phi\psi$ will destroy. Hence in this case $\phi\psi$ is indeterminate in the second degree.

Let
$$\phi = \begin{vmatrix} \phi_{11}, & \phi_{12}, & \phi_{13} \\ \phi_{21}, & \phi_{22}, & \phi_{23} \\ \phi_{31}, & \phi_{32}, & \phi_{33} \end{vmatrix},$$

i.e. $\phi\,(x_1 i_1 + x_2 i_2 + x_3 i_3) = (\phi_{11} x_1 + \phi_{12} x_2 + \phi_{13} x_3)\, i_1 + \dots$

say $\phi\,(ix) = \Sigma\Sigma\phi_{hk} i_h x_k$.

Then if ϕ is indeterminate, $|\phi| = 0$, and the points $\phi_1 i$, $\phi_2 i$, $\phi_3 i$ are all on the axis. The axis is therefore

$$\left\| \begin{matrix} \phi_{11}, & \phi_{21}, & \phi_{31} \\ \phi_{12}, & \phi_{22}, & \phi_{32} \end{matrix} \right\|,$$

and the null-point is

$$\begin{vmatrix} i_1, & i_2, & i_3 \\ \phi_{11}, & \phi_{12}, & \phi_{13} \\ \phi_{21}, & \phi_{22}, & \phi_{23} \end{vmatrix}.$$

If then the null-point of ϕ is on the axis of ψ, we must have

$$0 = \begin{vmatrix} \phi_{11}\psi_{11} + \phi_{12}\psi_{21} + \phi_{13}\psi_{31}, & \phi_{11}\psi_{12} + \phi_{12}\psi_{22} + \phi_{13}\psi_{32} \\ \phi_{21}\psi_{11} + \phi_{22}\psi_{21} + \phi_{23}\psi_{31}, & \phi_{21}\psi_{12} + \phi_{22}\psi_{22} + \phi_{23}\psi_{32} \end{vmatrix},$$

i.e. the first minors of $|\phi| \times |\psi'|$ must vanish, the rows of $|\phi|$ being multiplied into the columns of $|\psi|$. If the null-point of ψ is on the axis of ϕ, the first minors of $|\phi'| \times |\psi|$ must vanish, the columns of $|\phi|$ being multiplied into the rows of $|\psi|$*. Hence it is possible for $\phi\psi$ to be indeterminate in the second degree, while $\psi\phi$ is so only in the first. But $\phi\psi$ and $\psi\phi$ may both be doubly indeterminate without being equivalent. If however the null-point of each matrix is a united point on the axis of the other, and the remaining united points coincide at the intersection of the axes, then ϕ, ψ are commutative and their product is indeterminate in the second order.

In general two matrices are commutative when and only when they have the same united points. For let λ be a united point of ψ and not of ϕ, and suppose ϕ alters it to λ'. Then $\phi\psi$ will change λ to λ', but $\psi\phi$ will not do this unless λ' be another united point of ψ. Now there can be no cycle of changes, because the powers of ϕ have the same united points as ϕ. Hence, &c. Q.E.D.†

From this Cayley's theorem follows, because any set of n quantities $g_1 \dots g_n$ can be expressed linearly in terms of n other sets, $1, \phi, \phi^2 \dots \phi^{n-1}$. Consequently any matrix commutative with ϕ is of the form

$$a + b\phi + \dots + k\phi^{n-1}.$$

To form a matrix which shall have three given points for united points. Let the points be l, m, n; viz. $l = l_1 i_1 + l_2 i_2 + l_3 i_3$,

* [Observe that the rows of the unaccented symbol are multiplied into the columns of the accented symbol, the accents being used only for the purpose of marking this distinction. C.]

† [The argument seems to be, if ϕ and ψ are commutative, then $\phi\lambda = \lambda'$ a united point of ψ, and in like manner $\phi^2\lambda$, $\phi^3\lambda$, &c. are all of them united points of ψ, and being thus finite in number, there is some power $\phi^m\lambda$ which is $= \lambda$: viz. λ is a united point of ϕ^m, and therefore a united point of ϕ. C.]

$m = \&c.$, $n = \&c.$ Then $(lmn)\, i_1 = (m_2 n_3)\, l + (n_2 l_3)\, m + (l_2 m_3)\, n$, etc. Thus $(lmn)\, \phi i_1 = \lambda\, (m_2 n_3)\, l + \mu\, (n_2 l_3)\, m + \nu\, (l_2 m_3)\, n$ and

$$(lmn)\, \phi\, (xi) = (xmn)\, \lambda l + (xnl)\, \mu m + (xlm)\, \nu n$$

$$= \{(xmn)\, \lambda l_1 + (xnl)\, \mu m_1 + (xlm)\, \nu n_1\}\, i_1 + \ldots$$

$$\therefore\ (lmn)\, \phi_{hk} = \lambda l_h \frac{\partial\, (lmn)}{\partial l_k} + \mu m_h \frac{\partial\, (lmn)}{\partial m_k} + \nu n_h \frac{\partial\, (lmn)}{\partial n_k},$$

$$\phi = \begin{pmatrix} \Lambda_1, & \mathrm{M}_1, & \mathrm{N}_1 \\ \Lambda_2, & \mathrm{M}_2, & \mathrm{N}_2 \\ \Lambda_3, & \mathrm{M}_3, & \mathrm{N}_3 \end{pmatrix} \between \begin{pmatrix} \lambda l_1, & \lambda l_2, & \lambda l_3 \\ \mu m_1, & \mu m_2, & \mu m_3 \\ \nu n_1, & \nu n_2, & \nu n_3 \end{pmatrix}.$$

[Some lines, inserted apparently by way of verification, are omitted. The value obtained for the matrix ϕ is incorrect: we ought to have

$$(\phi \between l_1, l_2, l_3) = \lambda\, (l_1, l_2, l_3),$$
$$(\phi \between m_1, m_2, m_3) = \mu\, (m_1, m_2, m_3),$$
$$(\phi \between n_1, n_2, n_3) = \nu\, (n_1, n_2, n_3),$$

where ϕ is the required matrix

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix},$$

and (λ, μ, ν) are arbitrary constants.

There are thus nine linear equations for the determination of the nine coefficients of ϕ, and the solution is contained in the formula

$$\phi = \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} . \begin{pmatrix} \lambda, & 0, & 0 \\ 0, & \mu, & 0 \\ 0, & 0, & \nu \end{pmatrix} . \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix}^{-1}.$$

In fact starting from this formula, and substituting for ϕ its signification we have

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} . \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} = \begin{pmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{pmatrix} . \begin{pmatrix} \lambda, & 0, & 0 \\ 0, & \mu, & 0 \\ 0, & 0, & \nu \end{pmatrix},$$

or forming the products of the two pairs of matrices

$$\begin{array}{l|ccc} & (l_1, l_2, l_3), & (m_1, m_2, m_3), & (n_1, n_2, n_3) \\ \hline (a\ ,\ b\ ,\ c\ ) & ,, & ,, & ,, \\ (a', b', c') & ,, & ,, & ,, \\ (a'', b'', c'') & ,, & ,, & ,, \end{array}$$

$$= \begin{array}{l|ccc} & (\lambda, 0, 0), & (0, \mu, 0), & (0, 0, \nu), \\ \hline (l_1, m_1, n_1) & ,, & ,, & ,, \\ (l_2, m_2, n_2) & ,, & ,, & ,, \\ (l_3, m_3, n_3) & ,, & ,, & ,, \end{array}$$

that is

$$(a\ ,\ b\ ,\ c \between l_1, l_2, l_3) = (l_1, m_1, n_1 \between \lambda, 0, 0), = l_1 \lambda,$$
$$(a', b', c' \between\ \ ,,\ \ ) = (l_2, m_2, n_2 \between\ \ ,,\ \ ), = l_2 \lambda,$$
$$(a'', b'', c'' \between\ \ ,,\ \ ) = (l_3, m_3, n_3 \between\ \ ,,\ \ ), = l_3 \lambda,$$

and these are precisely the equations $(\phi \between l_1, l_2, l_3) = \lambda\, (l_1, l_2, l_3)$, &c. which were to be satisfied. C.]

*XXXVI.

ON TRICIRCULAR SEXTICS.

The focal properties of n-circular curves of order $2n$ are most easily studied by inverting their plane in regard to an external point; the plane then becomes a sphere, and the inverse curve is the general intersection of this sphere by a surface of the n^{th} order, so that the problem is reduced to the study of the intersection of a quadric with a surface of order n in its relations with a certain plane section of the quadric. The method is equivalent to either of the two following:—

(1) The method of representation of a unicursal surface upon a plane used by Clebsch and Cremona. Given a quadric surface (sphere), its plane sections are unicursal curves involving three variables, any two of which intersect in two points; they may therefore be fitly represented by plane conics passing through two fixed points (circles). The two fixed points represent two generators on the surface, and the line joining them represents the point of intersection of the generators; these constitute the exceptional system in the surface and in the plane. A section of the quadric surface by a surface of order n is represented by a plane curve order $2n$ passing n times through each of the fixed points; because the curve on the surface is met n times by each of the special generators; and conversely a curve of order $2n$ in the plane is in general represented by a curve of order $4n$ on the surface, but when the plane curve passes n times through each of the fixed points, the two special generators are each thrown off n times from the curve on the surface; this makes a reduction $2n$ in the order, and there remains only a curve of order $2n$ meeting every generator in n points, that is

to say, the intersection of the quadric by an n^{thic} surface. A tricircular sextic is in this way regarded as the representation on a plane of a quadricubic curve.

(2) The method of circular co-ordinates. If we write

$$\begin{aligned} X &= x^2 + y^2 + 2a_1x + 2b_1y + c_1, \\ Y &= x^2 + y^2 + 2a_2x + 2b_2y + c_2, \\ Z &= x^2 + y^2 + 2a_3x + 2b_3y + c_3, \\ W &= x^2 + y^2 + 2a_4x + 2b_4y + c_4, \end{aligned}$$

then the equation of every circle can be put into the form $lX + mY + nZ + sW = 0$. The quantities $XYZW$ satisfy identically a homogeneous equation of the second order, which may be written as follows, (XY) meaning the cosine of the angle of intersection of the circles X and Y: viz.,

$$\begin{vmatrix} 1, & (XY), & (XZ), & (XW), & Xr_1^{-1} \\ (YX), & 1, & (YZ), & (YW), & Yr_2^{-1} \\ (ZX), & (ZY), & 1, & (ZW), & Zr_3^{-1} \\ (WX), & (WY), & (WZ), & 1, & Wr_4^{-1} \\ Xr_1^{-1}, & Yr_2^{-1}, & Zr_3^{-1}, & Wr_4^{-1}, & 0 \end{vmatrix} = 0$$

where r_1, r_2, r_3, r_4 are the radii of the four circles. If they cut orthogonally this reduces to

$$X^2r_1^{-2} + Y^2r_2^{-2} + Z^2r_3^{-2} + W^2r_4^{-2} = 0.$$

In any case we may write this relation $\Omega_2 = 0$. An n-circular curve of order $2n$ may always be represented by an equation of order n in $XYZW$, say $U_n = 0$, and this in an infinity of ways. For if $U_n = 0$ represent any curve, $U_n + K_{n-2}\Omega_2 = 0$ will represent the *same* curve, whatever quantic of order $n-2$ in X, Y, Z, W is represented by K_{n-2}. But regarding $U_n = 0$, $\Omega_2 = 0$, since they contain four variables, as equations to surfaces, this amounts to saying that we have only to pay attention to their curve of intersection. So that this method is merely a translation into analytical language of the former one. It will however exhibit the results in a convenient form; for if we succeed in drawing through the curve of intersection some surface of the n^{th} order whose equation is of a simple form, this amounts to finding an equation for the plane curve of order

$2n$, which is the same as that of the surfaces except that what there meant the distance of a point from a fixed plane will here be interpreted to mean the squared tangent to a fixed circle.

The chief problem is to find the series of doubly tangent circles to the curve, among which the foci are included as special cases. They represent sections of the quadric surface by planes doubly tangent to the curve of intersection, and the foci correspond to those points of the quadric which are touched by such planes. Our problem is therefore, to study the torse which is the envelope of planes doubly tangent to the curve of intersection of a quadric with an n^{thic} surface, and especially the common tangent planes to their torse and the quadric.

The reciprocal problem is that of the determination of the focal curve of a surface of the n^{th} class. In elliptic space this curve is the nodal curve of the torse enveloped by planes tangent to the surface and a certain proper quadric, called the absolute; in parabolic space the absolute is a plane conic, and as this is part of the nodal curve (it is in fact an n-tuple curve on the torse) the degree of the focal curve is reduced. For some purposes it is convenient to consider the problem in its original form, and for others in the reciprocal form.

In any case, however, it is quite sufficiently complicated for tricircular sextics and surfaces of the third class; and I have only attempted it in this case of $n = 3$.

II.

A quadricubic curve, the intersection of a quadric with a cubic surface, when looked at from an arbitrary point in space, appears to be a sextic curve with six nodes and no cusps. It could only appear to have a cusp if the arbitrary point of observation were on the torse generated by tangents to the curve. To prove that it has six apparent nodes, we have only to remember that it is a point-for-point representative of a plane sextic with two triple points, and is therefore of

deficiency 4. The same thing appears by considering the cone standing on the curve and converging to a point of it. This is of the fifth order and has two nodal lines, namely the generators of the quadric through the point. The tangent cone, then, drawn through an arbitrary point of space, being of the sixth order with six nodal lines and no cusps, is of class $18 = 30 - 12$; and this is also the class of the tricircular sextic. In the latter case the number of tangents which can be drawn to the curve from one of the triple points is $12 = 18 - 6$, 2 tangents being swallowed up by each of the 3 branches through the triple point. Consequently the number of foci is 144, of which only 12 are real.

Returning to the quadricubic curve, we see that the tangent cone from an arbitrary point must have $36, = 6 \times 12 - 6 \times 6$ inflexions, and $96, = \frac{1}{2}(18 \times 17 - 3 \times 36 - 6)$ double tangents. The former number is the class of the torse traced out by tangents to the curve, and the latter is the class of the torse enveloped by planes doubly tangent. In the reciprocal problem, this latter number is the order of the nodal curve of the torse generated by common tangent planes to a quadric and a surface of the third class; that is to say, *the focal curve of a surface of the third class is of order* 96.

*XXXVII.

ON BESSEL'S FUNCTIONS.

1. I CONSIDER the function

$$f(x) = 1 + x + \frac{x^2}{(\Pi 2)^2} + \dots + \frac{x^k}{(\Pi k)^2} + \dots \textit{ ad inf.};$$

the series is evidently one-valued and convergent for all values real or complex of the variable x.

2. The n^{th} derived function of $f(x)$ is

$$f_n(x) = \frac{1}{\Pi n} + \frac{x}{\Pi(n+1)} + \frac{x^2}{\Pi(n+2).\Pi 2} + \dots + \frac{x^k}{\Pi(n+k).\Pi k} + \dots \textit{ ad inf.};$$

and if we integrate n times from 0 to x we obtain

$$f_{-n}(x) = \frac{x^n}{\Pi n} + \frac{x^{n+1}}{\Pi(n+1)} + \frac{x^{n+2}}{\Pi(n+2).\Pi 2} + \dots + \frac{x^{n+k}}{\Pi(n+k).\Pi k} + \dots = x^n f_n(x).$$

3. Hence we derive the differential equations

$$\partial_x^{2n}\{x^n f_n(x)\} = f_n(x),$$
$$\partial_x\{x^{n+1}\partial_x f_n(x)\} = x^n f_n(x).$$

4. We may generalize the definitions of (2) so as to obtain a value of $f_n(x)$ for all values of n, if we take Πn to mean $\Gamma(n+1)$, according to Gauss's notation. The differential equations of (3) will hold good in the general case.

5. By multiplying together the series for ϵ^{xy} and $\epsilon^{\frac{x}{y}}$, as follows:

$$1 + xy + \frac{x^2y^2}{\Pi 2} + \frac{x^3y^3}{\Pi 3} + \frac{x^4y^4}{\Pi 4} +$$
$$+$$
$$xy^{-1} + x^2 + \frac{x^3y}{\Pi 2} + \frac{x^4y^2}{\Pi 3} + \frac{x^5y^3}{\Pi 4} +$$
$$+$$
$$\frac{x^2y^{-2}}{\Pi 2} + \frac{x^3y^{-1}}{\Pi 2} + \frac{x^4}{(\Pi 2)^2} + \frac{x^5y}{\Pi 3.\Pi 2} + \frac{x^6y^2}{\Pi 4.\Pi 2} +$$
$$+$$
$$\frac{x^3y^{-3}}{\Pi 3} + \frac{x^4y^{-2}}{\Pi 3} + \frac{x^5y^{-1}}{\Pi 3.\Pi 2} + \frac{x^6}{(\Pi 3)^2} + \frac{x^7y}{\Pi 4.\Pi 3} +$$
$$+\frac{x^4y^{-4}}{\Pi 4} + \frac{x^5y^{-3}}{\Pi 4} + \frac{x^6y^{-2}}{\Pi 4.\Pi 2} + \frac{x^7y^{-1}}{\Pi 4.\Pi 3} + \frac{x^8}{(\Pi 4)^2} +$$

we find

$$e^{x\left(y+\frac{1}{y}\right)} = f(x^2) + \left(y + \frac{1}{y}\right).xf_1(x^2) + \left(y^2 + \frac{1}{y^2}\right).x^2f_2(x^2) + \dots$$
$$+ \left(y^k + \frac{1}{y^k}\right).x^kf_k(x^2) + \dots$$

6. In this formula write $-ix$ for x, and iy for y; then

$$e^{x\left(y-\frac{1}{y}\right)} = f(-x^2) + \left(y - \frac{1}{y}\right).xf_1(-x^2) + \left(y^2 + \frac{1}{y^2}\right).x^2f_2(-x^2) - \dots$$
$$+ \left\{y^k + \left(-\frac{1}{y}\right)^k\right\} x^k.f_k(-x^2) + \dots$$

a result which may also be obtained by direct multiplication.

7. In the equation of (5) and (6), put $y = e^{i\phi}$; thus we obtain

$$e^{2x\cos\phi} = f(x^2) + 2\cos\phi.xf_1(x^2) + 2\cos 2\phi.x^2f_2(x^2) + \dots$$
$$+ 2\cos k\phi.x^kf_k(x^2) + \dots,$$
$$e^{2ix\sin\phi} = f(-x^2) + 2i\sin\phi.xf_1(-x^2) + 2\cos 2\phi x^2f_2(-x^2) + \dots$$
$$+ 2i\sin(2k-1)\phi.x^{2k-1}f_{2k-1}(-x^2) + 2\cos 2k\phi.x^{2k}f_{2k}(-x^2) + \dots$$

and separating real and imaginary parts of the last equation,

$$\cos(2x\sin\phi) = f(-x^2) + 2\cos 2\phi \,.\, x^2 f_2(-x^2) + \ldots$$
$$+ 2\cos 2k\phi \, x^{2k} f_{2k}(-x^2) + \ldots$$
$$\sin(2x\sin\phi) = 2\sin\phi \,.\, x f_1(-x^2) + 2\sin 3\phi \,.\, x^3 f_3(-x^2) + \ldots$$
$$+ 2\sin(2k-1)\phi \,.\, x^{2k-1} f_{2k-1}(-x^2) + \ldots$$

8. Hence by Fourier's theorem we derive the definite integrals,

$$\int_0^\pi e^{2x\cos\phi}\cos n\phi d\phi = \pi x^n f_n(x^2),$$

$$\int_0^\pi \cos(2x\sin\phi)\cos 2n\phi d\phi = \pi x^{2n} f_{2n}(-x^2),$$

$$\int_0^\pi \cos(2x\sin\phi)\cos(2n+1)\phi d\phi = 0,$$

$$\int_0^\pi \sin(2x\sin\phi)\sin 2n\phi d\phi = 0,$$

$$\int_0^\pi \sin(2x\sin\phi)\sin(2n+1)\phi d\phi = \pi x^{2n+1} f_{2n+1}(-x^2),$$

and by addition of the last set of four

$$\int_0^\pi \cos(2x\sin\phi - n\phi)\, d\phi = \pi x^n f_n(-x^2).$$

[The above paper owed its origin to a communication made by Lord Rayleigh to the Mathematical Society (January 10th, 1878, *Proceedings*, Vol. IX. pp. 61—64). A few days after the meeting Prof. Clifford told me that he had a short note to lay before the Society, but I heard nothing further on the matter. Amongst the MSS. were two papers on the subject, the one printed above, the other entitled "Note on Bessel's and Laplace's functions." The former has been inserted in its entirety on Prof. Cayley's recommendation: I give an extract or two in this place from the latter. It commences—"Lord Rayleigh's interesting paper 'On the relation between the functions of Laplace and Bessel' has led me to examine whether a certain simple expression which I had found for Bessel's functions could not be extended to those of Laplace. It appears that the two functions may be derived by a very simple transformation from the exponential and binomial series respectively, and that the passage from one to the other is in fact equivalent to the well-known passage from $\left(1+\frac{x}{n}\right)^n$ to ϵ^x, when n is made indefinitely great."

He then points out that

$$J_n(2x) = x^n f_n(-x^2),$$

that the equation

$$\partial_x f_n(x) = f_{n+1}(x)$$

gives the formula for the fluxion of a Bessel's function, and derives the relation between three consecutive Bessel's functions from the identity

$$f_{m-1} - m f_m + x^2 f_{m+1} = 0.$$

From the equations for $\epsilon^{2x\cos\phi}$, $\epsilon^{2x\sin\phi}$, $\epsilon^{2ix\cos\phi}$ and $\epsilon^{2ix\sin\phi}$ he derives the first equation in 8 of the preceding paper. The remaining two pages are fragmentary, the only clear result being, that taking

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta P) + n(n+1)P = 0$$

and assuming

$$z = \tan^2 \tfrac{1}{2}\theta, \quad u = P \div \cos^{2n}\tfrac{1}{2}\theta,$$

he obtains

$$(1+z)^2 z\partial_z^2 u + \{(1+z)^2 - 2nz(1+z)\}\partial_z u + \{n(n+1) - n\}(1+z)u = 0,$$

or

$$(1+z)z\partial_z^2 u + (1+z-2nz)\partial_z u + n^2 u = 0 \quad \ldots\ldots\ldots\ldots (1).$$

This is satisfied, as Prof. Cayley points out, by

$$u = 1 - \left(\frac{n}{1}\right)^2 z + \left(\frac{n \,.\, n-1}{1 \,.\, 2}\right)^2 z^2 - \ldots,$$

viz. the resulting expression for P is

$$P = \cos^{2n}\tfrac{1}{2}\theta\left\{1 - \left(\frac{n}{1}\right)^2 \tan^2\tfrac{1}{2}\theta + \left(\frac{n \,.\, n-1}{1 \,.\, 2}\right)^2 \tan^4\tfrac{1}{2}\theta - \ldots\right\},$$

which is a known formula, cf. Todhunter, *Functions of Laplace, Lamé and Bessel*, p. 11.]

*XXXVIII.

ON GROUPS OF PERIODIC FUNCTIONS.

I. INTRODUCTION.

THE simplest periodic functions are the circular and elliptic functions; these may be regarded as built up out of exponential and θ-functions, and the latter again as built up of circular functions. Let us write *

$$\kappa u = e^u + e^{-u},$$

then $$\vartheta u = \Sigma_0^\infty q^{nn} \kappa(nu) = \Sigma_{-\infty}^{+\infty} e \,|\, n^2 Q + nu,$$

where $$q = e^Q.$$

In the same way if we write

$$\kappa(u, v) = e^u \vartheta(v + a) + e^{-u} \vartheta(v - a),$$

we may form the function

$$\vartheta(u, v) = \Sigma_0^\infty p^{mm} \kappa(mu, v) = \Sigma\Sigma_{-\infty}^{+\infty} e \,|\, m^2 P + n^2 Q + 2mna + mu + nv,$$

where it is to be understood that $\kappa(mu, v)$ means $e^{mu} \vartheta(v + ma) +$. By proceeding in this manner, we may form an ascending series of functions κ, ϑ, of an increasing number of variables. Let $m_1, m_2, \ldots m_p$ be p whole numbers, and let

$$\phi m = \Sigma Q_i m_i^2 + 2\Sigma Q_{ij} m_i m_j;$$

then we may form a function ϑ of the p arguments $u_1, u_2, \ldots u_p$ by adding together all the exponentials of the form

$$e^{\phi(m) + 2\Sigma mu},$$

* Rosenhain's Method, *Mem. Div. Sav.* t. XI.

obtained by giving to the whole numbers m all possible values from $-\infty$ to $+\infty$. That is, we shall write

$$\vartheta(u_1, u_2, \ldots u_p) = \Sigma^p e \,|\, \phi(m) + 2\Sigma mu \,|,$$

from this we may form a function κ with another argument u_0 as follows,

$$\kappa(u_0; u_1, u_2, \ldots u_p) = e^{2u_0} \vartheta(u_1 + a_1, u_2 + a_2, \ldots u_p + a_p) + e^{-2u_0} \vartheta(u_1 - a_1, \ldots u_p - a_p).$$

This being so, the functions ϑ and κ are unaltered if we increase any of the arguments u by any multiple of πi; so that the functions ϑ are p-fold periodic*, the functions κ are $(p + 1)$ fold periodic.

In any one term of the function ϑ let us increase the number m_i by unity. The alteration in $\phi(m)$ is $\partial_{m_i} \phi(m) +$, which is

$$2 Q_i m_i + 2\Sigma Q_{ij} m_j + Q_i,$$

the increase in $2\Sigma mu$ is simply $2u_i$. Thus the same effect will be produced if we increase every argument u_κ by the quantity $Q_{i\kappa}$, and then add $2u_i + Q_i$. Now the change from m_i to $m_i + 1$ merely passes from one term to another in the summation, and does not alter the function ϑ. Hence we have

$$\vartheta(u_1 \ldots u_p) = e^{2u_i + Q_i} \vartheta(u_1 + Q_{i1}, \ldots u_p + Q_{ip}).$$

Thus for simultaneous addition of one row of the quantities Q to the arguments, ϑ is quasi-periodic; it reproduces itself multiplied by an exponential factor.

The exponent in each term of ϑ, namely,

$$\phi(m) + 2\Sigma mu,$$

may be reduced to the form of a sum of squares of linear functions of the (m), less a quantity of the second order in the u. That is to say, we may have

$$\phi(m) + 2\Sigma mu = \Sigma_i (v_i + 2\Sigma m_j B_{ij})^2 - \Sigma v^2,$$

where the v are linear functions of the u. If then we multiply the ϑ-function by an exponential $e^{\Sigma v^2}$, we shall obtain the con-

* [Periodic as to these periods of the arguments u. C.]

venient form considered by Göpel, in which I now replace the letters v by the u;

$$G(u_1 \dots u_p) = \Sigma^p e \,|\, \Sigma_i (u_i + 2\Sigma m_j B_{ij})^2 \,|.$$

It is clear that this is unaltered if we simultaneously increase the arguments $u_1 \dots u_p$ by the quantities $2B_{1j} \dots 2B_{pj}$, where j may be any of the numbers 1, 2, p. There is also a set of quantities A in regard to which the function is quasi-periodic, or reproduces itself with an exponential factor. In regard to those quantities the quotient of G by $e \,|\, \Sigma u^2 \,|$ is actually periodic; viz., if we write

$$G(u_1 \dots u_p) = e \,|\, \Sigma u^2 \,|\, \Theta(u_1 \dots u_p),$$

then we shall have

$$\Theta(u_1 + 2A_{1j}, \dots u_p + 2A_{pj}) = \Theta(u_1 \dots u_p),$$

provided that

$$4\Sigma_j A_{ij} B_{ij} = \pi i, \quad 4\Sigma_j A_{ij} B_{kj} = 0,$$

and these equations suffice to determine the quantities A. We have then

$$\begin{aligned} G(u_1 + 2A_{1j} \dots u_p + 2A_{pj}) &= e \,|\, \Sigma_i (u_i + 2A_{ij})^2 \,|\, \Theta(u_1 \dots u_p) \\ &= e \,|\, 4\Sigma_i A_{ij}^2 + 4\Sigma_i u_i A_{ij} \,|\, G(u_1 \dots u_p). \end{aligned}$$

We shall now leave out in all cases the first suffix, writing for instance $G(u + A_j + B_k)$ instead of

$$G(u_1 + A_{1j} + B_{1k}, \dots u_p + A_{pj} + B_{pk}),$$

and so in similar cases.

II.

A linear function of the quantities A, B appertaining to a particular argument u, the coefficients being either 0 or 1, is called a *quadrant*, e.g. for u_1 a certain quadrant is

$$A_{11} + A_{12} + B_{12} + B_{13} + B_{1p}.$$

Since for every argument there are p A's and p B's, the whole number of quadrants is 2^{2p}, if 0 be included among them. We have now to consider the 2^{2p} functions $G(u + X)$, where X

is any quadrant. The quantities A, B themselves may be distinguished as *prime* or *elementary* quadrants.

If the prime quadrants of u are disposed in two rows, thus

$$A_1, A_2, A_3, \dots A_p,$$
$$B_1, B_2, B_3, \dots B_p,$$

they may be considered as forming the p *pairs* A_1B_1, A_2B_2,... A_pB_p. Thus every quadrant will be composed of a certain number of pairs, a certain number of single A's and a certain number of single B's.

If the number of pairs in a quadrant X is odd, then $G(u+X)$ is an odd function of the u, and $G(X) = 0$.

Let*

$$X = a_1A_1 + \dots + a_pA_p + b_1B_1 + \dots + b_pB_p,$$

where the a and the b are each of them either 0 or 1. We have

$$\begin{aligned} G(u + X) &= \Sigma e \,|\, \Sigma (u + X + 2\Sigma mB)^2 \,| \\ &= \Sigma e \,|\, \Sigma \{u + \Sigma aA + \Sigma (2m + b) B\}^2 \,|, \\ G(-u + X) = G(u - X) &= \Sigma e \,|\, \Sigma \{u - \Sigma aA + \Sigma (2m + b) B\}^2 \,| \\ &\qquad (m + 1 \text{ written for } m). \end{aligned}$$

Each exponent on the right is the same in these two expressions, except for the term $2\Sigma_k a_i b_i A_{ki} B_{ki}$ in the first, and $-2\Sigma_k a_i b_i A_{ki} B_{ki}$ in the second. The difference is $4a_i b_i \Sigma_k A_{ki} B_{ki} = a_i b_i \pi i$. Hence we have

$$e^{-4\Sigma Aau}\, G(u + X) = e \,|\, \pi i \Sigma a_i b_i \,|\, G(-u + X),$$

and therefore $e^{-2\Sigma Aau}\, G(u + X)$ is an odd or even function of u according as $\Sigma a_i b_i$ is odd or even.

The number of odd functions is $2^{p-1}(2^p - 1)$. We have to make

$$a_1 b_1 + a_2 b_2 + \dots a_p b_p = 2n + 1,$$

where the a, b are either 0 or 1. Let k of the a be zero, the remaining $p - k$, unity; then the b belonging to the first k may be either 0 or 1, which gives 2^k combinations.

* Clebsch and Gordan, p. 260.

The sum of the remaining $p-k$ b's must be odd, so that the last is determined when all the others are chosen, which may be done in 2^{p-k-1} ways; thus we have 2^{p-1} systems of the b. But the k a's which are zero may be chosen in $\frac{{}^*[p]^k}{\lfloor k}$ ways; and k may have all values from 0 to $p-1$. Hence the whole number of odd functions is

$$2^{p-1}\left(1+p+\frac{{}^*[p]^2}{2}+\frac{1}{\lfloor 3}{}^*[p]^3+\ldots+p\right)=2^{p-1}(2^p-1). \quad \text{Q.E.D.}$$

It follows that the number of even functions is $2^{p-1}(2^p+1)$.

Let X, Y, Z be any three quadrants, and consider the functions

$$G(u+Y+Z),\ G(u+Z+X),\ G(u+X+Y).$$

If they are all even, or if one is even and two odd, then each is of the same character as the product of the other two. But if they are all odd, or if one is odd and two even, then the fluxion of each is of the same character as the product of the other two. We shall now shew that

$$G'(u+X+Y)\,Gu-G(u+X+Y)\,G'u$$

may be expressed as the sum of a number of terms such as

$$G(u+X+Z),\ G(u+Y+Z).$$

Since Gu is an even function, it follows from what has just been said that of the three functions

$$G(u+Y+Z),\ G(u+Z+X),\ G(u+X+Y),$$

either all must be odd, or one odd and two even.

To this end it is important to consider the function which is derived from Gu by adding together the squares of all the terms; namely, we write

$$Fu=\Sigma e\,|\,2\Sigma_i\,(u_i+2\Sigma m_j B_{ij})^2\,|.$$

If now in $G(u+X+Z)$ we select the term involving the numbers m, and multiply it by the term in $G(u+Y+Z)$ involving the numbers n, the product is

$$e\,|\,\Sigma\,(u+X+Z+2\Sigma mB)^2+\Sigma\,(u+Y+Z+2\Sigma nB)^2\,|.$$

* [Auctoris.]

Now let $\mu_i=m_i+n_i,\ \nu_i=m_i-n_i;$

then we have

$$\mu^2+\nu^2=2(m^2+n^2),\ \mu\mu'+\nu\nu'=2(mm'+nn'),$$

and consequently

$$(u+X+Z+2\Sigma mB)^2+(u+Y+Z+2\Sigma nB)^2$$

= twice square of half sum + twice square of half difference

$$=2(u+\tfrac12X+\tfrac12Y+Z+\Sigma\mu B)^2+2(\tfrac12X-\tfrac12Y+\Sigma\nu B)^2.$$

In this expression each corresponding pair of μ, ν must be either both even or both odd. We may put separately the B's belonging to the odd ones; let $\mathfrak{B}$ be written for the sum of them, viz. $\mathfrak{B}$ is the sum of any selection of the B's; then $\Sigma\mu B$ may be written $\mathfrak{B}+2\Sigma m'B$ and $\Sigma\nu B=\mathfrak{B}+2\Sigma n'B$, where the m' and n' are now independent. Thus we shall have

$$G(u+X+Z)\,G(u+Y+Z)$$
$$=\Sigma F(u+\tfrac12X+\tfrac12Y+Z+\mathfrak{B})\,F(\tfrac12X-\tfrac12Y+\mathfrak{B}),$$

where the possible number of terms on the right is the number of different selections $\mathfrak{B}$, viz., 2^p.

We must however investigate the effect of adding half-quadrants to the arguments of the functions F. We have, namely,

$$F(u+\tfrac12X)=\Sigma e\,|\,2\Sigma\,\{u+\tfrac12\Sigma aA+\Sigma(2m+\tfrac12b)\,.\,B\}^2\,|$$
$$F(-u+\tfrac12X)=F(u-\tfrac12X)$$
$$=\Sigma e\,|\,2\Sigma\,\{u-\tfrac12\Sigma aA+\Sigma(2m-\tfrac12b)\,.\,B\}^2\,|$$
$$=\Sigma e\,|\,2\Sigma\,\{u-\tfrac12\Sigma aA+B_x+\Sigma(2m+\tfrac12b)\,.\,B\}^2\,|.$$
$$(B_x=\Sigma bB.)$$

But

$$F(u+B_x+\tfrac12X)=\Sigma e\,|\,2\Sigma\,\{u+\tfrac12\Sigma aA+B_x+\Sigma(2m+\tfrac12b)\,.\,B\}^2\,|,$$

the difference of these exponents consists of a term $4\Sigma aAu$, and a term $4\Sigma a\,.\,(4m+b)\,AB=\Sigma a\,.\,(4m+b)\,\pi i$. This therefore gives rise to a factor $e\,|\,\Sigma a\,.\,(4m+b)\,.\,\pi i\,|$ which is ±1 according as Σab is even or odd, i.e., according as $e^{-2\Sigma aAu}\,G(u+X)$ is even or odd. Thus

$$F(u-\tfrac12X)=\pm\,e^{-4\Sigma Aau}\,F(u+B_x+\tfrac12X),$$

according to the character of X.

* XXXIX.

THEORY OF MARKS OF MULTIPLE THETA-FUNCTIONS.

[A mark (p) is a symbol $\begin{pmatrix} \alpha, & \beta, & \gamma \ldots \\ \alpha', & \beta', & \gamma' \ldots \end{pmatrix}$, composed of the $2p$ integers $\alpha, \beta, \gamma \ldots \alpha', \beta', \gamma' \ldots$, distinguished only as even or odd integers; and the mark is even or odd according as $\alpha\alpha' + \beta\beta' + \gamma\gamma' \ldots$ is even or odd. C.]

NUMBER of marks (p) is 2^{2p}.

Number of odd marks $(p+1)$ may be calculated from number [of odd marks] p. Let N_p, E_p be numbers of odd and even marks for p; $N_p + E_p = 2^{2p}$. We may prefix to each of these odd marks ${0 \atop 0}$, ${0 \atop 1}$ or ${1 \atop 0}$, and to each of the even marks ${1 \atop 1}$, to get an odd mark for $p+1$. Therefore

$$N_{p+1} = 3N_p + E_p = 2^{2p} + 2N_p$$
$$= 2^{2p} + 2 \,.\, 2^{2(p-1)} + 2^2 \,.\, 2^{2(p-2)} + \ldots + 2^p = 2^p\,(2^{p+1} - 1);$$

and consequently $N_p = 2^{p-1}\,(2^p - 1)$. These numbers are 1, 6, 28, 120, ...

Any mark (p) except 0 may be divided into two odd marks in N_{p-1} ways. For let it begin with ${a \atop b}$. Take any odd mark $(p-1)$ for the places after the first, and add it to those places of the given mark. If the result is even, we must divide ${a \atop b}$ into an even and an odd part; if odd, we must divide it into two even parts. The former can be done in one way and one

only, unless ${a \atop b} = {0 \atop 0}$; viz. the parts are ${a+1 \atop b+1}$ and ${1 \atop 1}$. The latter can be done in one way only, viz. ${a \atop b} = {0 \atop 0} + {a \atop b}$ or ${1 \atop 1} = {1 \atop 0} + {0 \atop 1}$; but here we may attach either of the even parts to either of the odd succeeding parts, and we have consequently *two* divisions of the given mark corresponding to the *two* odd parts. Hence if ${b \atop a}$ is not $= {0 \atop 0}$, the number of divisions is exactly N_{p-1}. If it is, we may select any other place of the given mark, which is not $= {0 \atop 0}$.

Thus every mark g, except 0, is the sum of each of N_{p-1} pairs of odd marks. The $2N_{p-1}$ odd marks form the group g. We may include the group 0, consisting of all the odd marks. Then every odd mark occurs in N_p groups.

Every two odd marks l, m are common to $2N_{p-1}$ groups. A group to which l belongs is of the form $l+n$, where n is odd; if m belongs to this, $l+n = m+p$, and therefore $l+m = n+p$, or n, p form a pair of the group $l+m$. There are two groups for each such pair, except l, m; adding to these the group 0, we have $2N_{p-1}$.

To find the odd marks common to two groups, g, h, let $g+h = l+m$, where l, m are odd. Then

$$g+l = h+m, \quad g+m = h+l.$$

Now $(g+l)\,.\,(g'+l') + (h+l)\,.\,(h'+l') \equiv gh' + g'h$
$$\equiv gg' + hh' + (g+h)\,(g'+h').$$

If this $\equiv 1$, then $g+l$, $g+m$ are one even and one odd; therefore of every pair in the group $g+h$, one belongs to the group g and one to the group h; and the odd marks which make up these pairs in g, h are identical. In this case, therefore, every two of the groups g, h, $g+h$ have N_{p-1} marks in common, no two of which form a pair. For this, *one* or *all* of the marks g, h, $g+h$ must be odd. If however $gh' + g'h \equiv 0$, so that *none* or *two* of g, h, $g+h$ are odd, then $g+l$, $g+m$ are both even or both odd. Now in this case the three groups g, h, $g+h$ contain every odd mark between them. For

$$\begin{aligned}&(g+l)(g'+l')+(h+l)(h'+l')\\&=gg'+hh'+(g+h)l'+(g'+h')l\\&=(g+h+l)(g'+h'+l')+gh'+g'h+ll'.\end{aligned}$$

Therefore of $g+l$, $h+l$, $g+h+l$, either *one* is odd or *all* are odd. Let then each group have x marks in common with the other two, and y to itself. Then

$$x+3y=N_p,\quad x+y=2N_{p-1};$$

so that
$$y=\tfrac{1}{2}N_p-N_{p-1}=2^{p-2}(2^p-2^{p-1})=2^{2p-3},$$
$$x=2^{p-1}(2^{p-1}-1)-2^{2p-3}=2^{2p-3}-2^{p-1}=2^{p-1}(2^{p-2}-1)=4N_{p-2}.$$

These common marks are distributed in sets of 4, l, m, n, p, so that

$$g=l+n=m+p,\quad h=l+p=m+n,\quad g+h=l+m=n+p.$$

Relations of two groups.

Two groups g, h stand in one of two possible relations to each other. If they have N_{p-1} odd marks in common, no two of which are paired in either group, they shall be called *near* groups; if they have in common N_{p-2} sets of four marks, each set forming two pairs in each group, they shall be called *far* groups. The sum of two groups is near to or far from both of them, according as the groups are near or far. If they are near, no marks are common to the two groups and their sum; if they are far, the N_{p-2} sets occur in the sum divided into pairs in the only remaining way.

Hence, l, m, n being odd marks, the groups $l+m$, $l+n$ are near if l, which is common to them, does not belong to their sum, or if $l+m+n$ is even; far when $l+m+n$ is odd.

Relations of three groups.

Let f and g be two near groups, and h another group near to f but not equal to $f+g$; then if h be near to g, $f+h$ will be far from g, and *vice versâ*. The group h cannot contain all the marks common to f and g, for then $f+g$ would have absolutely no marks in common with h, which is impossible. Let then a be a mark common to f and g but not to h, and let $f=a+\alpha_1$, $g=a+\alpha_2$. Then if h is near to g, α_1 and α_2 both occur in h, but not paired, because $\alpha_1+\alpha_2=f+g$, which is not h by hypothesis. If $h=\alpha_1+\beta_2=\alpha_2+\beta_1$, then $a+h=f+\beta_2=g+\beta_1$, and therefore $f+\beta_1=g+\beta_2$, $=b$, suppose. Now b is odd, because either α_2 or β_1 must occur in the group f, and α_2 does not. Hence we have these equations,

$$\begin{aligned}f&=a+\alpha_1 & g&=a+\alpha_2 & h&=\alpha_1+\beta_2 & f+h&=a+\beta_2\\&=b+\beta_1 & &=b+\beta_2 & &=\alpha_2+\beta_1 & &=b+\alpha_2,\end{aligned}$$

from which it appears at once that $f+h$ is far from g, as was to be proved.

It follows that one-half of the groups which are near to f are also near to g, the complementary half (counting $f+h$ complementary to h) being far from g. We shall now consider more closely those groups which are near both to f and to g.

We observe that two pairs of the group h are obtained by crossing the constituents from two pairs of the groups f, g having in common the marks a, b. These marks shall be called a *cross-pair* of h; neither of them is contained in h. Since every cross-pair gives rise to a set of four marks common to the far groups g and $f+h$, and *vice versâ*, there are N_{p-2} cross-pairs. Every mark common to f and g either belongs to a cross-pair or occurs in h. Hence there are $N_{p-1}-2N_{p-2}$, $=2^{2p-4}$ marks common to f, g, h.

Relations of four groups.

If j be another group near to f, g and h, it contains one mark and one only out of every cross-pair of h. If it did not contain a or b, it must contain $\alpha_1, \beta_1, \alpha_2, \beta_2$, and therefore be far from h; if it contained both a and b, it must contain none of these four, with the same result. For the same reason h must contain one and one only out of every cross-pair of j. Hence of the marks common to f, g, there are N_{p-2} occurring neither in h nor j, and N_{p-2} occurring in each but not in the other. Thus $2^{2p-4} - N_{p-2}$ are common to all four; these are also common to $f+g+h+k$.

The system $fghj$ is formed as follows. Given f and g near to one another, we find h near to both; then it has N_{p-2} cross-pairs and 2^{2p-4} marks common with f and g. Let a belong to one of the cross-pairs, and c be one of the common marks; then a group formed to have ac for a cross-pair will be near to h. For it will contain α_1 and α_2 separated from each other, and associated with marks which do not occur in h. Let j be this group; we have seen that it has $2^{2p-4} - N_{p-2}$ marks common with fgh.

Now let α be one of these common marks; then a group k formed to have ad for a cross-pair will be near to h and j. All those N_{p-2} marks which are common to the cross-pairs of i and j will belong separately to the N_{p-2} cross-pairs of k. For if pq be a cross-pair of h, and pr the corresponding one of j, then k cannot have the cross-pair qr, or it would be $=f+g+h+j$, which we do not suppose; hence it must have a cross-pair ps. Then s must occur in the groups h and j. Consequently the number of marks common to $fghjk$ is $2^{2p-4} - 2N_{p-2}$. These are also common to $f+g+h+j$, $f+g+j+k$, $f+g+h+k$.

If j be near to f and g and far from h, then of the two marks in every cross-pair of h it must contain neither or both. If it contains a, for example, it cannot contain α_1 or α_2 and therefore not β_1 or β_2, so that it must contain b. Being near to f and g it is far from $f+g$, so that the three groups $f+g$, h, j are far from one another.

We may now find the groups to which three odd marks l, m, n are common. If g be one of these, $g+l$ is common to the groups g, $l+m$, $l+n$. We have therefore to find the marks which are common to the groups $l+m$, $l+n$. If

$$\begin{aligned} 1 &\equiv (l+m)(l'+n') + (l'+m')(l+n), \\ &\equiv ln' + l'n + mn' + m'n + lm' + l'm, \\ &\equiv (l+m+n)(l'+m'+n') + ll' + mm' + nn', \end{aligned}$$

i.e. if $l+m+n$ is even, then there are N_{p-1} marks common to the groups $l+m$, $l+n$; and by adding l to each of these, we obtain the N_{p-1} groups to which l, m, n are common. If however $l+m+n$ is odd, there are $4N_{p-2}$ marks common to the groups $l+m$, $l+n$, and consequently $4N_{p-2}$ groups to which the three marks l, m, n are common. In the former case, no two of the N_{p-1} marks common to $l+m$, $l+n$ are paired in either of them; let $l+g$, $l+h$ be any two of these, then $g+h$ is not equal to $l+m$, or $l+n$, or $m+n$; hence $l+g$, $m+g$, $n+g$ are different from $l+h$, $m+h$, $n+h$. Changing now the notation, let l_1, m_1, n_1 be three odd marks whose sum is even, and let $0, g_2, g_3 \dots g_n$, $(n = N_{p-1})$ be the groups to which they are common. Also let $g_r = l_1 + l_r = m_1 + m_r = n_1 + n_r$; then all the marks l, m, n are different. Moreover the three odd marks l_r, m_r, n_r are common to the N_{p-1} groups $l_r + l_s, = m_r + m_s, = n_r + n_s$. These three marks are identical, because

$$l_1 + l_r = m_1 + m_r = n_1 + n_r,$$

and $l_1 + l_s = m_1 + m_s = n_1 + n_s$. Hence $l_r + m_r + n_r$ is even.

All the groups to which three odd marks are common shall be called a *set*. Then we have N_{p-1} sets 1, 2, 3 ..., all containing the group 0, and any two of them r, s having also in common the group $(rs) = l_r + l_s =$ etc.

Consider now the case of three odd marks l, m, n, whose sum is odd, and which are therefore common to $4N_{p-2}$ groups. The marks common to 0, $l+m$, $l+n$ arrange themselves in sets of fours, $pqrs$, so that

$$l+m = p+q = r+s, \quad l+n = p+r = q+s.$$

Hence the groups containing l, m, n arrange themselves in sets of fours,

$$\begin{array}{r|r|r|r} f = l+p & g = l+q & h = l+r & k = l+s \\ = m+q & = m+p & = m+s & = m+r \\ = n+r & = n+s & = n+p & = n+q \\ = 0+s & = 0+r & = 0+q & = 0+p, \end{array}$$

so that

$$l+m = f+g = h+k, \quad l+n = f+h = g+k, \quad m+n = g+h = f+k.$$

Let $l_1 + m_1 + n_1 + p_1 = 0$, so that we have to deal with four odd marks whose sum is zero; these will be common to N_{p-2} tetrads of groups f_i, g_i, h_i, k_i; and if

$$f_i = l_1 + l_i = m_1 + m_i = n_1 + n_i = p_1 + p_i,$$

we shall have N_{p-2} tetrads l, m, n, p. The order of any tetrad must only be altered by a bifid substitution; such tetrad has therefore four forms; by adding two tetrads of odd marks in various forms we obtain always the same tetrad of groups f, g, h, k. Any such tetrad of odd marks (i) is common to the N_{p-2} tetrads of groups (i, j).

We pass now to the consideration of three groups g, h, k which are not such that $g+h+k=0$, but which are such that every two of them are *contiguous*, i.e. they have N_{p-1} marks in common; and we propose to determine the number of marks common to all three groups. Let a be a mark common to g and h but not to k, so that $g=a+\alpha_1$, $h=a+\alpha_2$; then α_1 and α_2 both occur in k, but not paired, because $\alpha_1+\alpha_2=g+h$, which is not k by hypothesis. Hence $g+h$ and k are non-contiguous groups, since two marks which are paired in $g+h$ occur in k. If $k=\alpha_1+\beta_1=\alpha_2+\beta_2$, then $a+k=g+\beta_1=h+\beta_2$, and therefore $g+\beta_2=h+\beta_1=b$, suppose. Thus $g=b+\beta_2$, $h=b+\beta_1$, so that the six marks $a, b, \alpha_1, \beta_1, \alpha_2, \beta_2$ are distributed as follows:

$$\begin{array}{llll} g=a+\alpha_1 & h=a+\alpha_2 & k=\alpha_1+\beta_1 & g+h=\alpha_1+\alpha_2 \\ \quad =b+\beta_2 & \quad =b+\beta_1 & \quad =\alpha_2+\beta_2 & \qquad =\beta_1+\beta_2. \end{array}$$

Since $g+h$ and k are non-contiguous, there are N_{p-2} such sets of six marks. If c be a mark common to g and h, not belonging to any of these sets, it must also belong to k; and consequently we must have $g=c+\gamma_1$, $h=c+\gamma_2$, $k=c+\gamma_3$. Of such marks c there are $N_{p-1}-2N_{p-2}=N_{p-2}+E_{p-2}=2^{2p-4}$. This is then the number of marks common to three such groups.

Hence four odd marks l, m, n, r, such that $l+m$, $l+n$, $l+r$ are contiguous two and two but their sum not zero, are common to 2^{2p-4} groups; and we have a theory of 2^{2p-4} such sets of four odd marks, any two sets being common to one group besides the group 0.

Let λ be another mark common to the groups $l+m$, $l+n$, $l+r$, and let $l+m=\lambda+\mu$, $l+n=\lambda+\nu$, $l+r=\lambda+\rho$; then λ, μ, ν, ρ form another such set. If g is a group containing l, m, n, r, then $g+l$ belongs to the groups $l+m$, $l+n$, $l+r$, that is, to the groups $\lambda+\mu$, $\lambda+\nu$, $\lambda+\rho$. To find the groups g, then, we must take the marks p belonging to the groups $l+m$, $l+n$, $l+r$ and add them to l. Similarly, to find the groups containing λ, μ, ν, ρ, we must add the same marks p to λ. Suppose two of them p, q are such that $p+l=q+\lambda$, then this group $p+l$ contains all the eight marks $l, m, n, r, \lambda, \mu, \nu, \rho$. So also will the group $p+\lambda$, $=q+l$. For this, we must have $p+q=l+\lambda$; we have therefore to find the number of marks common to the four groups $l+m$, $l+n$, $l+r$, $l+\lambda$. The last is non-contiguous to the first three; we know that

$$l+\lambda=m+\mu=n+\nu=r+\rho.$$

Let $g+h+k=0$, and $g=a+\alpha=b+\beta=c+\gamma$,

$$h=a+\beta=b+\alpha,\quad k=a+b=\alpha+\beta;$$

then if $a+b+c$ is odd, c occurs in the group k; it therefore occurs among the $4N_{p-2}$ marks common to g and k. Thus the marks in g are such that the sum of two from the same tetrad, not paired, and one belonging to no tetrad is always even.

Hence if we take two unpaired marks a, b, and a third c not belonging to the group $a+b$, then a will not belong to the group $b+c$, etc. How many are common to the groups g, $a+b$, $a+c$? If d is common to them, we have

$$g=d+\delta=a+\alpha=b+\beta=c+\gamma,$$

$$d+\epsilon=a+b=\alpha+\beta,\quad d+\theta=a+c=\alpha+\gamma.$$

Common to g, $a+b$ are $4N_{p-2}$ including $ab\alpha\beta$,

„ g, $a+c$ „ $4N_{p-2}$ „ $ac\alpha\gamma$.

Suppose that

$$g=a+\alpha=b+\beta,\ =c+\gamma=d+\delta,\ =e+\epsilon=f+\phi,$$
$$h=a+b=\alpha+\beta,\ =c+d=\gamma+\delta,\ =e+f=\epsilon+\phi,$$
$$k=a+\beta=\alpha+b,\ =c+\delta=\gamma+d,\ =e+\phi=\epsilon+f,$$

a mark may be divided in 2^{2p-1} ways into two marks, namely N_{p-1} ways into two odd marks

$$2^{p-2}(2^{p-1}-1)=2^{2p-3}-2^{p-2};$$

E_{p-1}, $=2^{2p-3}+2^{p-2}$, ways into two even marks; and consequently 2^{2p-2} ways into one odd and one even mark. Thus every mark g can be divided in 2^{2p-2} ways into h and k so that their groups shall have N_{p-2} marks common.

This is the same number as if we were permitted to select the marks to be made common in any way we liked from $2p-2$ of the pairs in g.

Thus, $p=3$, we may divide every mark in 32 ways, and therefore in 16 ways, so that three groups shall have six marks common.

Every [triple] θ-function has 28 zero-values; every value annuls 28 θ-functions.

Every two θ-functions have 12 zero-values in common; every two values annul 12 θ-functions.

The three functions $0, g, h$ have 6 zero-values in common, if

$$gg'+hh'+(g+g')(h+h')\equiv 1,$$

or if $gh'+g'h\equiv 1$ (mod. 2), in this case also the values $0, g, h$ annul 6 θ-functions.

This is true of the functions g, h, k, if

$$(g+h)(g'+k')+(g'+h')(g+k)$$

or

$$hk'+h'k+kg'+k'g+gh'+g'h\equiv 1,$$

or

$$(g+h+k)(g'+h'+k')-gg'-hh'-kk'\equiv 1.$$

If however this quantity is even, the three functions have only 4 zero-values in common.

Six θ-functions having 3 common zeros shall be called a *set*. If the set includes θ_0, the 3 zeros are all odd; whence it appears that their sum must be even. Hence they belong to one or more of Weber's "groups," so that no two form a "pair." Every even mark may be expressed in 56 ways as the sum of three different odd marks; thus the number of sets including θ_0

is 56×36; and generally this is the number of sets including any given mark.

The number of sets including θ_0 and θ_a is $160=8.20$. For the six pairs in the group (a) give us 20 triads of pairs, and each of these gives eight triads of odd marks constituting a set.

If $0, g, h$; $0, h, k$; $0, k, g$ have each 6 zeros in common, there are 4 zeros common to $0, g, h, k$, unless $g+h+k=0$, when there are none common (Weber).

Suppose then lmn, lnr, lrm to be zeros belonging to three sets which include θ_0. Then

$$0,\ l+m,\ l+n;\quad 0,\ l+n,\ l+r;\quad 0,\ l+r,\ l+m$$

are zeros belonging to three sets which include θ_l. Now

$$l+m+l+n+l+r=l+m+n+r;$$

consequently this must not be zero, if the three sets are to have four θs common. Of these three sets, then, any two can only have in common such θs as belong also to the third.

Let $g=l+l'=m+m'=n+n'$; then the sets lmn, $l'm'n'$ have no θ in common except θ_0 and θ_g. For two groups g, h have 4 or 6 zeros in common, the latter only when no two of them form a pair in either group. Let now h be a mark of the set lmn, so that $h=l+\alpha=m+\beta=n+\gamma$. Then either $\alpha\beta\gamma$ must be different from $l'm'n'$; or, suppose $\alpha=m'$, then $\beta=l'$, and $\gamma=l+m'+n=l'+m+n$. But $l+m'+n$ is necessarily even, so that $m'l'\gamma$ is not a set. Starting then from the set $l_1l_2l_3$, let the marks of it be $0, a, b, c, f, g$; then

$$a=l+m,\quad b=l+n,\quad c=l+p,\quad f=l+q,\quad g=l+r;$$

i.e.

$$a=l_1+m_1=l_2+m_2=l_3+m_3.$$

Any two of the six sets l, m, n, p, q, r have one mark in common besides 0. For (e.g.)

$$p_1+q_1=c+f=p_2+q_2=p_3+q_3.$$

$$\begin{aligned} g&=a+\alpha_1 & h&=a+\alpha_2 & k&=\alpha_1+\beta_1\\ &=b+\beta_2 & &=b+\beta_1 & &=\alpha_2+\beta_2.\end{aligned}$$

If g, h, k are contiguous but sum not zero, the same is true of g, $g+h$, $g+k$, etc.

$$\begin{array}{lll} h+k=a+\beta_2 & k+g=a+\beta_1 & g+h=\alpha_1+\alpha_2 \\ \quad =b+\alpha_1 & \quad =b+\alpha_2 & \quad =\beta_1+\beta_2. \end{array}$$

$$\left.\begin{array}{r} g+h+k=\alpha_2+\beta_1 \\ =a+b \\ =\alpha_1+\beta_2 \end{array}\right\} \text{ is non-contiguous to all six.}$$

If $l+m$, $l+n$, $l+r$ are contiguous two and two but their sum not zero, then $m+n$, $m+r$ are also contiguous. Hence the sum of every three of the four marks l, m, n, r is even.

Now suppose a fourth group f, contiguous to g. If it is non-contiguous to h, then $f+g$ will be contiguous to both of them. Hence one-half of the groups contiguous to g are also contiguous to h. Suppose then that f is contiguous to g and h, then it is non-contiguous to $g+h$.

If the mark a is common to g and h but not to f,

$$g=a+\alpha_1, \quad h=a+\alpha_2,$$

then $f=\alpha_1+\gamma_1=\alpha_2+\gamma_2$, so that $g+\gamma_2=h+\gamma_1$, $=c$, suppose, where c is odd, because either α_2 or γ_2 must occur in the group g, and α_2 does not. Hence we have

$$\begin{array}{lllll} j=\beta_1+\gamma_2 & g=a+\alpha_1 & h=a+\alpha_2 & k=\alpha_1+\beta_1 & f=\alpha_1+\gamma_1 \\ \;=\beta_2+\gamma_1 & \;=b+\beta_2 & \;=b+\beta_1 & \;=\alpha_2+\beta_2 & \;=\alpha_2+\gamma_2 \\ \;=a+\alpha_3 & \;=c+\gamma_2 & \;=c+\gamma_1 & \;=c+\gamma_3 & \;=b+\beta_3. \\ & \;=d+\delta_2 & \;=d+\delta_1 & \;=d+\delta_3 & \end{array}$$

If therefore f is contiguous to k, γ_1 and γ_2 do not occur in k, and therefore c does; also b in f. But c cannot occur in f, or b in k.

To make a system of four groups, contiguous two and two, and having common marks; take two contiguous groups

$$g, \; =a+\alpha_1=b+\beta_1=c+\gamma_1, \text{ etc.,}$$

and

$$h, \; =a+\alpha_2=b+\beta_2=c+\gamma_2, \text{ etc.}$$

To these add the group k, $=\alpha_1+\beta_2=\alpha_2+\beta_1=$ etc., which does

not contain a or b. Select two marks, c, d, common to g, h, k, then the fourth group, if any, must be $\gamma_1+\delta_2=\gamma_2+\delta_1$, $=f$, suppose.

$$\begin{array}{lllll} j=\beta_1+\gamma_2 & g=a+\alpha_1 & h=a+\alpha_2 & k=\alpha_1+\beta_2 & f=\gamma_1+\delta_2 \\ \;=\beta_2+\gamma_1 & \;=b+\beta_1 & \;=b+\beta_2 & \;=\alpha_2+\beta_1 & \;=\gamma_2+\delta_1 \\ \;=a+\alpha_3 & \;=c+\gamma_1 & \;=c+\gamma_2 & \;=c+\gamma_3 & \\ & \;=d+\delta_1 & \;=d+\delta_2 & \;=d+\delta_3 & \end{array}$$

$$g+h=c+d+f.$$

f must contain either a or b but not both. Suppose a, then it contains β_1 and β_2.

f is non-contiguous to $h+k=\gamma_2+\gamma_3=\delta_2+\delta_3$.

Since f contains either a or b but not both, this case is the same as the last. f is linked with g, h by N_{p-2} cross-pairs. Out of each cross-pair of k, f must contain *one* mark but not both; thus there are N_{p-2} marks common to g and h which belong to cross-pairs both of k and f, and $2N_{p-2}$ belonging to cross-pairs of k and f separately. Hence the number of marks common to g, h, k, f is $2^{2p-4}-N_{p-2}$.

$$\begin{array}{l} j=\beta_2+\gamma_1=\beta_1+\gamma_2 \\ \;=a+\alpha_3 \end{array}$$

must not contain α_1, α_2, c, b, but must contain a.

$$f+g+h+j+k=0.$$

*XL.

ON THE DOUBLE THETA-FUNCTIONS.

THE θ-series of two variables u_1, u_2 is defined as follows:

$$\theta(u_1, u_2;\ a_{11}, a_{12}, a_{22}) = \sum_{n_1}\sum_{n_2} \epsilon\,|\, n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1 u_1 + 2n_2 u_2 \,|^*,$$

where the whole numbers n_1, n_2, in respect of which the summation takes place, may have all values from $-\infty$ to $+\infty$. The parameters a_{11}, a_{12}, a_{22} must be such that the real part of $n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22}$ is negative for all values of n_1, n_2. This is necessary and sufficient for the convergency of the series.

The function θ is unaltered if either of the arguments be increased by any multiple of πi; this follows from the fact that $\epsilon\,|\,2\pi i\,| = 1$. This is expressed by saying that u_1, u_2 have independently the period πi; or that taken together they have the two periods πi, 0 and 0, πi.

As the number n_1 may have all integer values, the function will not be altered if we write $n_1 + q_1$ for it (q_1 being any integer) and then give n_1 all integer values. Thus we find

$$\begin{aligned}\theta(u_1, u_2) &= \sum_{n_1}\sum_{n_2} \epsilon\,|\,(n_1+q_1)^2 a_{11} + 2(n_1+q_1) n_2 a_{12} + n_2^2 a_{22} + 2(n_1+q_1) u_1 + 2n_2 u_2\,| \\ &= \epsilon\,|\,q_1^2 a_{11} + 2q_1 u_1\,| \,.\, \sum_{n_1}\sum_{n_2} \epsilon\,|\,n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1(u_1 + q_1 a_{11}) + 2n_2(u_2 + q_1 a_{12})\,| \\ &= \epsilon\,|\,q_1^2 a_{11} + 2q_1 u_1\,|\,\theta(u_1 + q_1 a_{11}, u_2 + q_2 a_{12});\end{aligned}$$

* [Or as it might be written

$$\Sigma_{n1}\Sigma_{n2} \exp.\{(a_{11}, a_{12}, a_{22} \between n_1, n_2)^2 + 2n_1 u_1 + 2n_2 u_2\}.]$$

or, which is the same thing,

$$\theta(u_1 + q_1 a_{11}, u_2 + q_1 a_{12}) = \epsilon\,|-q_1^2 a_{11} - 2q_1 u_1\,|\,\theta(u_1, u_2).$$

In the same manner we may shew that

$$\theta(u_1 + q_2 a_{12}, u_2 + q_2 a_{22}) = \epsilon\,|-q_2^2 a_{22} - 2q_2 u_2\,|\,\theta(u_1, u_2).$$

From these equations it appears that when the arguments u_1, u_2 are simultaneously increased by the parameters a_{11}, a_{12} respectively, the function θ becomes multiplied by the exponential factor $\epsilon\,|-a_{11} - 2u_1\,|$; and that when they are simultaneously increased by a_{12}, a_{22} respectively, the θ is multiplied by $\epsilon\,|-a_{22} - 2u_2\,|$. On this account the arguments u_1, u_2 are said to have the *quasi-periods* a_{11}, a_{12} and a_{12}, a_{22}.

The effect of adding a *half*-period to either argument is to change the sign of certain terms in the series. Thus if we write $u_1 + \frac{1}{2}\pi i$ for u, each term becomes affected with the factor $\epsilon\,|\,n_1 \pi i\,|$, that is $(-)^{n_1}$. The series thus arising are conveniently distinguished by the numbers 0 or 1 affixed to the θ, as follows: $\phi(n_1, n_2)$ being for shortness written instead of

$$n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1 u_1 + 2n_2 u_2,$$

$$\theta(u_1 + \tfrac{1}{2}\pi i, u_2) = \Sigma\Sigma\,(-)^{n_1}\,\epsilon\,|\,\phi(n_1, n_2)\,| = \theta^{10}(u_1, u_2),$$

$$\theta(u_1, u_2 + \tfrac{1}{2}\pi i) = \Sigma\Sigma\,(-)^{n_2}\,\epsilon\,|\,\phi(n_1, n_2)\,| = \theta^{01}(u_1, u_2),$$

$$\theta(u_1 + \tfrac{1}{2}\pi i, u_2 + \tfrac{1}{2}\pi i) = \Sigma\Sigma\,(-)^{n_1+n_2}\,\epsilon\,|\,\phi(n_1, n_2) = \theta^{11}(u_1, u_2).$$

We may include these in the formula

$$\begin{aligned}\theta^{\alpha\beta}(u_1 + \tfrac{1}{2}\gamma\pi i, u_2 + \tfrac{1}{2}\delta\pi i) &= \Sigma\Sigma\,(-)^{(\alpha+\gamma)n_1 + (\beta+\delta)n_2}\,|\,\phi(n_1, n_2)\,| \\ &= \theta^{\alpha+\gamma,\,\beta+\delta}(u_1, u_2),\end{aligned}$$

where, in $\theta^{\alpha\beta}$, we must suppose 0 or 1 substituted for each of the whole numbers α, β according as it is even or odd.

The effect of adding a half quasi-period to the arguments is to add $\frac{1}{2}$ to one of the numbers n_1, n_2 and multiply the θ-function by an exponential factor. The new series thus arising are conveniently distinguished by 0 or 1 *suffixed* to the θ, as follows:

$$\begin{aligned}\theta(u_1 + \tfrac{1}{2}a_{11}, u_2 + \tfrac{1}{2}a_{12}) &= \epsilon\,|-\tfrac{1}{4}a_{11} - u_1\,| \,.\, \Sigma\Sigma\epsilon\,|\,\phi(n_1 + \tfrac{1}{2}, n_2)\,| \\ &= \epsilon\,|-\tfrac{1}{4}a_{11} - u_1\,| \,.\, \theta_{10}(u_1, u_2),\end{aligned}$$

$$\theta\left(u_1+\tfrac{1}{2}a_{12},\ u_2+\tfrac{1}{2}a_{22}\right)=\epsilon\left|-\tfrac{1}{4}a_{22}-u_2\right|.\ \Sigma\Sigma\epsilon\left|\phi\left(n_1,\ n_2+\tfrac{1}{2}\right)\right|$$
$$=\epsilon\left|-\tfrac{1}{4}a_{11}-u_1\right|.\ \theta_{01}(u_1,\ u_2),$$

$$\theta\left(u_1+\tfrac{1}{2}a_{11}+\tfrac{1}{2}a_{12},\ u_2+\tfrac{1}{2}a_{12}+\tfrac{1}{2}a_{22}\right)$$
$$=\epsilon\left|-\tfrac{1}{4}a_{11}-\tfrac{1}{2}a_{12}-\tfrac{1}{4}a_{22}-u_1-u_2\right|.\ \Sigma\Sigma\epsilon\left|\phi\left(n_1+\tfrac{1}{2},\ n_2+\tfrac{1}{2}\right)\right|$$
$$=\epsilon\left|-\tfrac{1}{4}a_{11}-\tfrac{1}{2}a_{12}-\tfrac{1}{4}a_{22}-u_1-u_2\right|.\ \theta_{11}(u_1,\ u_2).$$

These cases are included in the formula

$$\theta_{\alpha\beta}\left(u_1+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22}\right)$$
$$=\epsilon\left|-\phi\left(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta\right)\right|.\ \theta_{\alpha+\gamma,\ \beta+\delta}(u_1,\ u_2),$$

where, in the suffixes, we must suppose 0 or 1 written for each number according as it is even or odd.

In regard to the periods and quasi-periods of these θ, it is to be remarked that θ^{10}, θ^{01} and θ^{11} have each the period πi for each argument, like θ; but that besides acquiring the exponential factor, θ^{10} changes sign for addition of the quasi-period a_{11}, a_{12}, θ^{01} for addition of a_{12}, a_{22}, and θ^{11} for addition of either. On the other hand, θ_{10}, θ_{01}, and θ_{11} resemble θ in regard to the quasi-periods, but change sign when u_1, u_2, or either of them is increased by πi respectively. A general formula for these cases will be given presently.

If we add both half-periods and half-quasi-periods to the arguments, we obtain nine new θ-series, which together with the preceding are all defined in the following formula:

$$\theta\left(u_1+\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22}\right)$$
$$=\epsilon\left|-\phi\left(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta\right)\right|\Sigma\Sigma(-)^{\alpha n_1+\beta n_2}\epsilon\left|\phi\left(n_1+\tfrac{1}{2}\gamma,\ n_2+\tfrac{1}{2}\delta\right)\right|$$
$$=\epsilon\left|-\phi\left(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta\right)\right|\theta^{\alpha\beta}_{\gamma\delta}(u_1,\ u_2).$$

Here each of the letters α, β, γ, δ stands for either 0 or 1.

This formula may be generalized into the following, which contains the entire theory of the transformation of the θ into one another by addition of half-periods and half-quasi-periods to the arguments.

$$\theta^{ab}_{cd}\left(u_1+\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ u_2+\tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22}\right)$$
$$=\epsilon\left|\tfrac{1}{2}(\alpha c+\beta d)\pi i-\phi\left(\tfrac{1}{2}\gamma,\ \tfrac{1}{2}\delta\right)\right|\theta^{a+\alpha,\ b+\beta}_{c+\gamma,\ d+\delta}(u_1,\ u_2).$$

Here α, β, γ, δ may be any whole numbers, and the affixes and suffixes are to be understood as before.

Of the sixteen functions θ^{ab}_{cd}, six are odd functions of u_1, u_2, namely those for which $ac+bd$ is an odd number; and these of course vanish for the values $u_1=0$, $u_2=0$. The other ten are even functions, and it is easy by means of the preceding formulæ to assign for each the six pairs of values which make it zero. Namely

$$\theta^{ab}_{cd}\left(\tfrac{1}{2}\alpha\pi i+\tfrac{1}{2}\gamma a_{11}+\tfrac{1}{2}\delta a_{12},\ \tfrac{1}{2}\beta\pi i+\tfrac{1}{2}\gamma a_{12}+\tfrac{1}{2}\delta a_{22}\right)=0,$$

whenever $\qquad (a+\alpha)(c+\gamma)+(b+\beta)(d+\delta)$

is an odd number.

The Product-Theorem.

If we multiply together term by term two θ-series with different arguments, we shall obtain a quadruply infinite series, the exponent in each term of it being the sum of the exponents in those terms of the θ-series from which it is derived. Thus we shall have

$$\theta(u_1,\ u_2)\,\theta(v_1,\ v_2)=\Sigma\Sigma\epsilon\left|\phi(m_1,\ m_2)\right|.\ \Sigma\Sigma\epsilon\left|\psi(n_1,\ n_2)\right|$$
$$=\Sigma\Sigma\Sigma\Sigma\epsilon\left|\phi(m_1,\ m_2)+\psi(n_1,\ n_2)\right|,$$

where

$$\phi(m_1,\ m_2)=m_1^2a_{11}+2m_1m_2a_{12}+m_2^2a_{22}+2m_1u_1+2m_2u_2,$$
$$\psi(n_1,\ n_2)=n_1^2a_{11}+2n_1n_2a_{12}+n_2^2a_{22}+2n_1v_1+2n_2v_2.$$

Now if we write

$$m_1+n_1=p_1,\ m_1-n_1=q_1,$$
$$m_2+n_2=p_2,\ m_2-n_2=q_2,$$

we shall have

$$2(m_1^2+n_1^2)=p_1^2+q_1^2,\ 2(m_2^2+n_2^2)=p_2^2+q_2^2,$$
$$2(m_1m_2+n_1n_2)=p_1p_2+q_1q_2,$$

and therefore

$$\phi(m_1,\ m_2)+\psi(n_1,\ n_2)$$
$$=\tfrac{1}{2}(p_1^2+q_1^2)\,a_{11}+(p_1p_2+q_1q_2)\,a_{12}+\tfrac{1}{2}(p_2^2+q_2^2)\,a_{22}$$
$$+p_1(u_1+v_1)+q_1(u_1-v_1)+p_2(u_2+v_2)+q_2(u_2-v_2)$$
$$=\tfrac{1}{2}p_1^2a_{11}+p_1p_2a_{12}+\tfrac{1}{2}p_2^2a_{22}+p_1(u_1+v_1)+p_2(u_2+v_2)$$
$$+\tfrac{1}{2}q_1^2a_{11}+q_1q_2a_{12}+\tfrac{1}{2}q_2^2a_{22}+q_1(u_1-v_1)+q_2(u_2-v_2).$$

Now the p and q are not all numbers indiscriminately, but p_1 and q_1 must be both odd or both even, being the sum and difference of two integers, and so also p_2 and q_2 must be both odd or both even. In the quadruply infinite series, therefore, there are four kinds of terms; those in which the p, q are all odd, in which they are all even, in which p_1, q_1 are odd and p_2, q_2 even, and in which p_1, q_1 are even and p_2, q_2 odd. We shall sum these separately.

First let all four numbers be even; and let

$$p_1 = 2s_1, \quad p_2 = 2s_2, \quad q_1 = 2t_1, \quad q_2 = 2t_2,$$

then

$$\begin{aligned}\phi(m_1, m_2) + \psi(n_1, n_2) &= 2s_1^2 a_{11} + 4s_1 s_2 a_{12} + 2s_2^2 a_{22} + 2s_1(u_1+v_1) + 2s_2(u_2+v_2)\\ &+ 2t_1^2 a_{11} + 4t_1 t_2 a_{12} + 2t_2^2 a_{22} + 2t_1(u_1-v_1) + 2t_2(u_2-v_2).\end{aligned}$$

If we sum the exponential of this quantity for all integer values of s and t, we shall clearly obtain

$$\theta(u_1+v_1, u_2+v_2; 2a_{11}, 2a_{12}, 2a_{22}) \,.\, \theta(u_1-v_1, u_2-v_2; 2a_{11}, 2a_{12}, 2a_{22});$$

that is, the product of two θ-series whose parameters are the doubles of those we have been considering. To avoid the trouble of expressing these parameters in every case, it is convenient to denote such θ-series by a capital Θ; thus

$$\Theta(u_1, u_2) = \theta(u_1, u_2;\ 2a_{11}, 2a_{12}, 2a_{22}).$$

Next let p_1, q_1 be odd, while p_2, q_2 are even; and let

$$p_1 = 2s_1 + 1, \ p_2 = 2s_2, \ q_1 = 2t_1 + 1, \ q_2 = 2t_2,$$

then

$$\begin{aligned}\phi(m_1, m_2) + \psi(n_1, n_2) &= 2(s_1+\tfrac{1}{2})^2 a_{11} + 4(s_1+\tfrac{1}{2}) s_2 \,.\, a_{12} + 2s_2^2 a_{22} + 2(s_1+\tfrac{1}{2})(u_1+v_1)\\ &\qquad + 2s_2(u_2+v_2)\\ &+ 2(t_1+\tfrac{1}{2})^2 a_{11} + 4(t_1+\tfrac{1}{2}) t_2 \,.\, a_{12} + 2t_2^2 a_{22} + 2(t_1+\tfrac{1}{2})(u_1-v_1)\\ &\qquad + 2t_2(u_2-v_2),\end{aligned}$$

and the sum of the exponentials of this quantity for all integer values of s and t is clearly

$$\Theta_{10}(u_1+v_1, u_2+v_2)\,\Theta_{10}(u_1-v_1, u_2-v_2).$$

Similarly if p_1, q_1 be even and p_2, q_2 odd, we shall obtain

$$\Theta_{01}(u_1+v_1, u_2+v_2)\,\Theta_{01}(u_1-v_1, u_2-v_2),$$

and for all four numbers odd,

$$\Theta_{11}(u_1+v_1, u_2+v_2)\,\Theta_{11}(u_1-v_1, u_2-v_2).$$

The product $\theta(u_1, u_2)$, $\theta(v_1, v_2)$ is thus equal to the sum of these four products. To state the proposition with brevity and clearness, we shall mention only one of the two variables in each case, omitting the suffix; thus $\theta(u)$ will stand for $\theta(u_1, u_2)$, and $\Theta(u+v)$ for $\Theta(u_1+v_1, u_2+v_2)$. What we have proved, then, is that

$$\begin{aligned}\theta u.\theta v = &\ \ \Theta\ (u+v)\,\Theta\ (u-v)\\ &+\Theta_{10}(u+v)\,\Theta_{10}(u-v)\\ &+\Theta_{01}(u+v)\,\Theta_{01}(u-v)\\ &+\Theta_{11}(u+v)\,\Theta_{11}(u-v)\\ =&\ \Sigma\Theta_{ab}(u+v)\,\Theta_{ab}(u-v). \qquad (a, b = 0, 1).\end{aligned}$$

From this formula we may, by adding half-periods and half-quasi-periods to the arguments u, v, obtain an expression for any such product as $\theta_{cd}^{ab} u \,.\, \theta_{\gamma\delta}^{\alpha\beta} v$. The number of such distinct products is 136, including the theorem just stated; the general formula including them all will be subsequently examined. But the case in which $\alpha, \beta, \gamma, \delta = a, b, c, d$ admits of simple treatment and leads to some important consequences.

First, let us adopt the abbreviation

$\theta\left(u + \begin{matrix} ab \\ cd \end{matrix}\right)$ for

$$\theta(u_1 + \tfrac{1}{2}a\pi i + \tfrac{1}{2}ca_{11} + \tfrac{1}{2}da_{12},\ u_2 + \tfrac{1}{2}b\pi i + \tfrac{1}{2}ca_{12} + \tfrac{1}{2}da_{22}),$$

then we may write

$$\theta\left(u + \begin{matrix} ab \\ cd \end{matrix}\right) = \epsilon\,|-\phi(\tfrac{1}{2}c, \tfrac{1}{2}d)|\,\theta_{cd}^{ab}(u),$$

$$\theta\left(v + \begin{matrix} ab \\ cd \end{matrix}\right) = \epsilon\,|-\psi(\tfrac{1}{2}c, \tfrac{1}{2}d)|\,\theta_{cd}^{ab}(v),$$

$$\Theta\left(u + v + \begin{matrix} ab \\ cd \end{matrix}\right) = \epsilon\,|-\phi(\tfrac{1}{2}c, \tfrac{1}{2}\delta)|\,\Theta_{cd}^{ab}(u+v),$$

where

$$\phi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right) = \tfrac{1}{4}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + cu_1 + du_2,$$
$$\psi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right) = \tfrac{1}{4}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + cv_1 + dv_2,$$
$$\Phi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right) = \tfrac{1}{2}(c^2a_{11} + 2cda_{12} + d^2a_{22}) + c(u_1 + v_1) + d(u_2 + v_2)$$
$$= \phi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right) + \psi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right).$$

Hence by substitution we obtain*

$$\epsilon\left| -\phi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right) - \psi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right)\right| \theta^{ab}_{cd}(u)\, \theta^{ab}_{cd}(v)$$
$$= \Sigma\Theta_{pq}\left(u + v + \begin{matrix}2a, & 2b\\ c & d\end{matrix}\right)\Theta_{pq}(u - v)$$
$$= \Sigma\epsilon\left|(ap+bq)\pi i - \Phi\left(\tfrac{1}{2}c, \tfrac{1}{2}d\right)\right| \Theta_{p+c, q+d}(u+v)\, \Theta_{pq}(u-v),$$

and consequently

$$\theta^{ab}_{cd}(u)\, \theta^{ab}_{cd}(v) = \Sigma\, (-)^{ap+bq}\, \Theta_{p+c, q+d}(u+v)\, \Theta_{pq}(u-v) \qquad (p, q=0, 1).$$

In this equation write u for v; thus

$$\{\theta^{ab}_{cd}(u)\}^2 = \Sigma\, (-)^{ap+bq}\, \Theta_{p+c, q+d}(2u)\, \Theta_{pq}(0).$$

It appears from this result that the square of each of the θ-functions is a linear function of the four quantities

$$\Theta_{00}(2u),\ \Theta_{10}(2u),\ \Theta_{01}(2u),\ \Theta_{11}(2u);$$

the coefficients being the quantities

$$\Theta_{00}(0),\ \Theta_{10}(0),\ \Theta_{01}(0),\ \Theta_{11}(0).$$

It follows therefore that *a linear relation may be found between the squares of any five of the θ-functions.*

In certain cases, however, it may be shewn that such a relation holds between the squares of four of them. Just as each θ vanishes for each of six pairs of values of the arguments, so each such pair of values annuls six θ-functions. Consider four of these six having the same zero, and one other θ which is not annulled by that pair of values. By what we have just proved, there is a linear relation between the squares of these five; but this is impossible unless the coefficient of the fifth be zero, because we can give to the arguments such values that

* On the right-hand side it is to be observed that c, d are written instead of $2c$, $2d$, because of the double parameters of Θ.

the first four functions shall vanish, but not the fifth. It follows therefore that

If four θ-functions vanish for the same pair of values of the variables, their squares are connected by a linear relation.

For example, the four functions

$$\theta(u),\ \theta^{11}(u),\ \theta^{10}_{10}(u),\ \theta^{10}_{01}(u)$$

are all reduced to zero by the values $u = \begin{matrix}0, 1\\ 1, 1\end{matrix}$, that is to say, $u_1 = \tfrac{1}{2}a_{11} + \tfrac{1}{2}a_{12}$, $u_2 = \tfrac{1}{2}i\pi + \tfrac{1}{2}a_{12} + \tfrac{1}{2}a_{22}$. To find the coefficients in the linear relation, let us assume

$$x\theta(u)^2 + y\theta^{11}(u)^2 + z\theta^{10}_{10}(u)^2 + w\theta^{10}_{01}(u)^2 = 0;$$

then we have only to observe that every two of these functions have one other zero in common, so that by giving to the u this pair of values we can find the ratio of two of the coefficients. Thus for

$$u = \begin{matrix}10\\ 11\end{matrix}, \text{ we have } -z\theta_{01}(0)^2 + w\theta_{10}(0)^2 = 0,$$
$$u = \begin{matrix}11\\ 10\end{matrix}, \quad ,, \quad ,, \quad y\theta_{10}(0)^2 - z\theta^{01}(0)^2 = 0,$$
$$u = \begin{matrix}01\\ 00\end{matrix}, \quad ,, \quad ,, \quad x\theta^{01}(0)^2 + y\theta^{10}(0)^2 = 0,$$

by successive applications of the formula

$$\theta^{ab}_{cd}\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix} = \epsilon\left|\tfrac{1}{2}(\alpha c + \beta d)\pi i - \phi\left(\tfrac{1}{2}\gamma, \tfrac{1}{2}\delta\right)\right| \theta^{a+\alpha,\, b+\beta}_{c+\gamma,\, d+\delta}(0).$$

The result is that

$$-x : y : z : w = \theta^{10}(0)^2 : \theta^{01}(0)^2 : \theta_{10}(0)^2 : \theta_{01}(0)^2,$$

so that the sought relation is

$$-\theta^{10}(0)^2.\theta(u)^2 + \theta^{01}(0)^2.\theta^{11}(u)^2 + \theta_{10}(0)^2.\theta^{10}_{10}(u)^2 + \theta_{01}(0)^2.\theta^{10}_{01}(u)^2 = 0.$$

With a view to further investigation of the group of six functions θ^{11}, θ^{10}_{10}, θ^{10}_{01}, θ^{01}_{10}, θ^{01}_{01}, θ_{11}, whose characteristics are the sums in pairs of the four characteristics $\begin{matrix}10\\ 00\end{matrix}$, $\begin{matrix}01\\ 00\end{matrix}$, $\begin{matrix}00\\ 10\end{matrix}$, $\begin{matrix}00\\ 01\end{matrix}$, it will be useful to set down here three other relations which connect triads of their squares with $\theta(u)^2$. They are obtained in precisely the same manner as the one already written down.

$$-\theta^{10}(0)^2.\theta^{11}(u)^2+\theta^{01}(0)^2.\theta(u)^2-\theta_{10}(0)^2.\theta_{10}^{01}(u)^2-\theta_{01}(0)^2.\theta_{01}^{01}(u)^2=0,$$
$$-\theta^{10}(0)^2.\theta_{10}^{10}(u)^2+\theta^{01}(0)^2.\theta_{10}^{01}(u)^2-\theta_{10}(0)^2.\theta(u)^2+\theta_{01}(0)^2.\theta_{11}(u)^2=0,$$
$$-\theta^{10}(0)^2.\theta_{01}^{10}(u)^2+\theta^{01}(0)^2.\theta_{01}^{10}(u)^2-\theta_{10}(0)^2.\theta_{11}(u)^2+\theta_{01}(0)^2.\theta(u)^2=0.$$

These four equations, it will be observed, are not independent; for if they be multiplied respectively by $\theta^{10}(0)^2$, $\theta^{01}(0)^2$, $\theta_{10}(0)^2$ and $\theta_{01}(0)^2$, the result is $\theta(u)^2$ multiplied by

$$-\theta^{10}(0)^4+\theta^{01}(0)^4-\theta_{10}(0)^4+\theta_{01}(0)^4,$$

which vanishes, as may be proved by writing $u=\begin{matrix}10\\00\end{matrix}$ in the first equation.

Eliminating the constant multipliers, we obtain

$$\begin{vmatrix} -\theta^2, & +\theta^{11^2}, & +\theta_{10}^{10^2}, & +\theta_{01}^{10^2} \\ -\theta^{11^2}, & +\theta^2, & -\theta_{10}^{01^2}, & -\theta_{01}^{01^2} \\ -\theta_{10}^{10^2}, & +\theta_{10}^{01^2}, & -\theta^2, & +\theta_{11}^{\ 2} \\ -\theta_{01}^{10^2}, & +\theta_{01}^{01^2}, & -\theta_{11}^{\ 2}, & +\theta^2 \end{vmatrix}.$$

[Pages of MS. up to this point are numbered 1 to 15 in Clifford's own handwriting: then comes *the Fluxion-Theorem* not numbered, and some more pages "too incomplete for printing. C."]

The Fluxion-Theorem.

A θ-series may be differentiated in regard to either of the variables, or, as GÖPEL suggested, the question may be left open, if we write $\partial=x\partial_{u_1}+y\partial_{u_2}$, where x and y are quantities to be determined at the end of the investigation. This being so, we find

$$\theta(u).\partial\theta(v)-\theta(v).\partial\theta(u)=2\Sigma\Theta_{pq}(u+v).\partial\Theta_{pq}(u-v)\qquad(p,\,q=0,\,1),$$

by the same process as that which was used for the product-theorem. It is necessary for our subsequent investigation to

extend this theorem so that for $\theta(u)$ we may write $\theta_{cd}^{ab}(u)$. To this end we proceed as follows. We have

$$\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\theta_{cd}^{ab}(u),$$

therefore

$$\partial\theta\left(u+\begin{matrix}ab\\cd\end{matrix}\right)=\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\partial\theta_{cd}^{ab}(u)-\theta_{cd}^{ab}(u)\,\partial\phi\}.$$

Again,

$$\Theta_{pq}\left(u\pm v+\begin{matrix}a & b\\ \tfrac{1}{2}c, & \tfrac{1}{2}d\end{matrix}\right)=\epsilon\,|\,\tfrac{1}{2}(ap+bq)\,\pi i\,|\,\Theta_{pq}^{ab}\,(u\pm v+\tfrac{1}{2}c,\,\tfrac{1}{2}d),$$

therefore

$$\partial\Theta_{pq}\left(u-v+\begin{matrix}a & b\\ \tfrac{1}{2}c, & \tfrac{1}{2}d\end{matrix}\right)=\epsilon\,|\,\tfrac{1}{2}(ap+bq)\,\pi i\,|\,.\,\partial\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

By substitution of these values, the fluxion-theorem becomes when we write in it $u+\begin{matrix}ab\\cd\end{matrix}$ for u,

$$\epsilon\,|-\phi(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\theta_{cd}^{ab}(u)\,.\,\partial\theta(v)-\theta(v)\,\partial\theta_{cd}^{ab}(u)-\theta_{cd}^{ab}(u)\,\theta(v)\,\partial\phi\}$$
$$=2\Sigma\,(-)^{ap+bq}\,\Theta_{pq}^{ab}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)$$
$$(p,\,q=0,\,1).$$

But by the product-theorem we have also

$$\epsilon\,|-\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\theta_{cd}^{ab}(u)\,\theta(v)$$
$$=\Sigma\,(-)^{ap+bq}\,\Theta_{pq}^{ab}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

Moreover

$$2\partial\,.\,\epsilon^{\frac{1}{2}\phi}\,\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)$$
$$=\epsilon^{\frac{1}{2}\phi}\,\{2\partial\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)+\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\phi\},$$

therefore finally

$$\epsilon\,|-\tfrac{1}{2}\phi\,(\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,|\,\{\theta_{cd}^{ab}u\,.\,\partial\theta v-\theta v\,.\,\partial\theta_{cd}^{ab}u\}$$
$$=2\Sigma\,(-)^{ap+bq}\,\Theta_{pq}^{ab}(u+v+\tfrac{1}{2}c,\,\tfrac{1}{2}d)\,\partial\,.\,\epsilon^{\frac{1}{2}\phi}\,\Theta_{pq}^{ab}(u-v+\tfrac{1}{2}c,\,\tfrac{1}{2}d).$$

*XLI.

MOTION OF A SOLID IN ELLIPTIC SPACE*.

I.

THE *second moment* of a solid body in regard to a plane is the sum obtained by multiplying the mass of each particle into the squared sine of its distance from the plane and adding together the products thus formed. The body being referred to a quadrantal tetrahedron, let the equation of the plane be

$$\Sigma \xi x \equiv \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0,$$

then the coordinates of any particle p of mass P being $p_1 p_2 p_3 p_4$, the second moment of the plane is

$$\int P\,(\Sigma \xi p)^2 = k^2, \text{ suppose,}$$

it being assumed that $\Sigma p^2 = 1$, $\Sigma \xi^2 = 1$, always. Moreover, as only one body will be here considered, its mass may conveniently be taken as the unit of mass; so that the quantity k is to be called the swing-radius of the body in regard to the plane ξ.

We may write also

$$k^2 = (1, 1)\,\xi_1^2 + (2, 2)\,\xi_2^2 + \ldots$$
$$+ 2\,(1, 2)\,\xi_1 \xi_2 + \ldots$$

where $$(1, 1) = \int P p_1^2, \text{ etc.; } (1, 2) = \int P p_1 p_2, \text{ etc.}$$

* [At the Meeting of the *London Mathematical Society* held February 12, 1874, Prof. Clifford gave an account of a paper on "the Free motion of a Solid in Elliptic Space": I take XLI. to be this paper.]

The quantities (1, 1), (1, 2) are the moments and products of inertia of the body in respect of the coordinate planes.

Those planes whose second moment vanishes envelop a surface of the second order $k^2 = 0$ (the null-surface), which determines all the dynamical relations of the body. The six axes of this surface (being the edges of the self-conjugate tetrahedron common to it and the absolute) are called the principal axes of the body. We may for all dynamical purposes substitute for the body a system of four particles having appropriate masses placed at the vertices of any tetrahedron self-conjugate to the null-surface.

If two planes at right angles be drawn through any axis, having k_1, k_2 for their swing-radii, then $h^2 = k_1^2 + k_2^2$ is a constant independent of the orientation of the two planes, and h is called the swing-radius of the body in regard to that axis. Those axes whose swing-radii vanish are therefore such that the tangent planes drawn through them to the null-quadric are at right angles; or the lines are harmonically related to the null-quadric and the absolute. Hence, if the coordinates of an axis are λ_{12}, λ_{23}, etc., we shall have

$$h^2 = \{(1, 1) + (2, 2)\}\,\lambda_{12}^2 + \text{etc.} + 2\,(12)\,(\lambda_{31}\lambda_{23} - \lambda_{14}\lambda_{24})$$
$$+ \text{etc.,}$$

which is easily verified.

It must be observed that we have

$$(1, 1) + (2, 2) + (3, 3) + (4, 4) = 1,$$

which gives three relations among the coefficients in the expression for h^2.

Suppose now that the instantaneous velocity of the body is the twist

$$V = \alpha + \omega\beta = ix + jy + kz + \omega\,(iu + jv + kw).$$

Here x, y, z are the component rotations about axes through the origin, and u, v, w component velocities along them. Let a, b, c be the swing-radii of the body about the axes i, j, k; then $\sqrt{1 - a^2}$, $\sqrt{1 - b^2}$, $\sqrt{1 - c^2}$ will be the swing-radii about

ωi, ωj, ωk. Let also f^2, g^2, h^2 be the products of inertia in regard to the planes through the origin, l^2, m^2, n^2 the products of inertia of these with the polar of the origin. Supposing for the moment that $xu + yv + zw = 0$, the twist becomes a mere rotation, and we can find the swing-radius of its axis. Namely, it is d, where

$$\begin{aligned}-(\alpha+\omega\beta)^2 . d^2 = {} & a^2x^2+b^2y^2+c^2z^2+(1-a^2).u^2+(1-b^2).v^2+(1-c^2).w^2 \\ & +f^2(yz-vw)+g^2(zx-wu)+h^2(xy-uv) \\ & +l^2(yw-zv)+m^2(zu-xw)+n^2(xv-yu) = f_2(\alpha+\omega\beta).\end{aligned}$$

Now this expression is evidently twice the kinetic energy of the body, since $T(\alpha+\omega\beta)$ is the angular velocity about the axis. But it is easy to see that the same expression represents twice the kinetic energy in the general case also when $xu+yv+zw$ is not $=0$, and $\alpha+\omega\beta$ is a motor. For, referring it to its axes, we can express it as the sum of two polar rotors; viz.,

$$\begin{aligned}\alpha+\omega\beta = {} & \tfrac{1}{4}\{\xi U(\alpha+\beta)+\eta U(\alpha-\beta)\}\{T(\alpha+\beta)+T(\alpha-\beta)\} \\ & +\tfrac{1}{4}\{\xi U(\alpha+\beta)-\eta U(\alpha-\beta)\}\{T(\alpha+\beta)-T(\alpha-\beta)\} \\ = {} & A+\lambda\omega A, \text{ say; where if}\end{aligned}$$

$$A = ix+jy+kz+\omega(iu+jv+kw),$$

then $$\omega A = iu+jv+kw+\omega(ix+jy+kz),$$

and $$f_2(A+\lambda\omega A) = f_2(A)+\lambda^2 f_2(\omega A),$$

because $$xu+yv+zw = 0.$$

But these rotors A and ωA being polar, the velocities due to them are everywhere at right angles; so that the energy of their resultant is the sum of their energies; that is, it is represented by $f_2(A+\lambda\omega A)$.

Now let $\nabla = i\partial_x + j\partial_y + k\partial_z + \omega(i\partial_u + j\partial_v + k\partial_w) = \nabla_1 + \omega\nabla_2$, then, T being the kinetic energy, ∇T is a motor which we shall call the *momentum* of the moving body. Namely, it is

$$\begin{aligned}M = {} & (a^2x+h^2y+g^2z+n^2v-m^2w)\,i \\ & +(h^2x+b^2y+f^2z+l^2w-n^2u)\,j \\ & +(g^2x+f^2y+c^2z+m^2u-l^2v)\,k \\ & +\{(1-a^2).u-h^2y-g^2w-n^2y+m^2z\}\,\omega i \\ & +\{-h^2u+(1-b^2).v-f^2w-l^2z+n^2x\}\,\omega j \\ & +\{-g^2.u-f^2v+(1-b^2).w-m^2x+l^2y\}\,\omega k.\end{aligned}$$

If we write

$$\phi(ix+jy+kz) = \left(\begin{array}{ccc} a^2, & h^2, & g^2 \\ h^2, & b^2, & f^2 \\ g^2, & f^2, & c^2 \end{array}\right)\!(xyz)(ijk),$$

then the expression for M may be written

$$M = \phi\alpha + \omega(\iota-\phi)\beta + \sigma(\beta-\omega\alpha),$$

where $$\sigma = l^2 i + m^2 j + n^2 k,$$

or $$\begin{aligned}M & = \tfrac{1}{2}(\alpha+\omega\beta)+(\phi-\tfrac{1}{2})(\alpha-\omega\beta)-\omega\sigma(\alpha-\omega\beta) \\ & = \tfrac{1}{2}V+(\phi-\tfrac{1}{2}-\omega\sigma)(\alpha-\omega\beta) \\ & = \tfrac{1}{2}V+\chi(\alpha-\omega\beta),\end{aligned}$$

where $$\chi = \left(\begin{array}{ccc} a^2-\tfrac{1}{2}, & h^2-\omega n^2, & g^2+\omega m^2 \\ h^2+\omega n^2, & b^2-\tfrac{1}{2}, & f^2-\omega l^2 \\ g^2-\omega m^2, & f^2+\omega l^2, & c^2-\tfrac{1}{2} \end{array}\right).$$

The rate of variation of M is equal to the resultant impressed wrench; now in the case where the impressed forces have a potential P, this resultant wrench is $-\nabla P$. Hence the equation of motion is

$$\dot{M} + \nabla P = 0,$$

wherein M is understood to be expressed as a linear function of V. The kinetic energy T satisfies the equation

$$S.VM + 2T = 0.$$

II.

In the case where there are no forces, P is a constant, and we have

$$\dot{M} = 0,$$

or $$M = \text{a constant motor.}$$

Hence the expression for M as a linear function of V gives us six first integrals of the equation of motion. We have also $S.VM + 2T = 0$, which, now that T and M are constant, gives us a seventh relation among the components of V. This may, however, be deduced from the other six.

In this form the equations are not convenient.

The quantities a^2, b^2, c^2, etc. depend upon the position of the body in regard to the coordinate planes, and are therefore variable. It would be necessary to express them in terms of the position-motor of the body; i.e. the motor which would bring the body from its initial to its final position; then to express the velocity in terms of the variation of this motor (to which it is not equal); and to integrate the resulting equations. We can avoid all this by using moving axes*.

In the first place we have to determine the rates of change of the coordinates of any point p in terms of the twist-velocity $\alpha + \omega\beta$. These are

$$\begin{aligned} \dot{p}_1 &= \qquad\quad - zp_2 + yp_3 + up_4, \\ \dot{p}_2 &= \;\; zp_1 \qquad\; - xp_3 + vp_4, \\ \dot{p}_3 &= -yp_1 + xp_2 \qquad\;\, + wp_4, \\ \dot{p}_4 &= -up_1 - vp_2 - wp_3. \end{aligned}$$

From these we can find the rates of change of the quantities a^2, etc., namely,

$$\begin{aligned} \tfrac{1}{2}\frac{d(a^2)}{dt} &= \tfrac{1}{2}\frac{d}{dt}\int P(p_2^2 + p_3^2) = \int P(p_2\dot{p}_2 + p_3\dot{p}_3) \\ &= z\int Pp_1p_2 - y\int Pp_1p_3 + v\int Pp_2p_4 + w\int Pp_3p_4 \\ &= h^2z - g^2y + m^2v + n^2w, \end{aligned}$$

$$\begin{aligned} \frac{d(f^2)}{dt} &= \frac{d}{dt}\int Pp_2p_3 = \int P(p_2\dot{p}_3 + \dot{p}_2p_3) \\ &= x\int P(p_2^2 - p_3^2) + z\int Pp_1p_3 - y\int Pp_1p_2 \\ &\qquad\qquad + v\int Pp_3p_4 + w\int Pp_2p_4 \\ &= (c^2 - b^2)x + g^2z - h^2y + n^2v + m^2w, \end{aligned}$$

* [Cf. Fig. 52.]

$$\begin{aligned} \frac{d(l^2)}{dt} &= \frac{d}{dt}\int Pp_1p_4 = \int P(p_1\dot{p}_4 + \dot{p}_1p_4) \\ &= u\int P(p_4^2 - p_1^2) + y\int Pp_3p_4 - z\int Pp_2p_4 \\ &\qquad\qquad - v\int Pp_1p_2 - w\int Pp_1p_3 \\ &= (1 - b^2 - c^2)u + n^2y - m^2z - h^2v - g^2w, \end{aligned}$$

and from these the other six may be written down by symmetry.

Now let the moving axes be taken to be the principal axes of the body. Then all the products of inertia vanish, and we have

$$M = a^2i + b^2j + c^2k + \omega\{(1 - a^2)u + (1 - b^2)v + (1 - c^2)w\}$$

$$\tfrac{1}{2}\frac{d(a^2)}{dt} = 0,$$

$$\frac{d(f^2)}{dt} = (c^2 - b^2)x, \quad \frac{d(g^2)}{dt} = (a^2 - c^2)y, \quad \frac{d(h^2)}{dt} = (b^2 - a^2)z,$$

$$\frac{d(l^2)}{dt} = (1 - b^2 - c^2)u, \quad \frac{d(m^2)}{dt} = (1 - c^2 - a^2)v, \quad \frac{d(n^2)}{dt} = (1 - a^2 - b^2)w.$$

Now by differentiating the coefficients of i and ωi in the value of M on page 380, we find

$$\frac{d}{dt}(a^2x + h^2y + g^2z + n^2v - m^2w) = 0,$$

$$\frac{d}{dt}\{(1 - a^2).u - h^2v - g^2w - n^2y + m^2z\} = 0,$$

that is to say

$$\begin{aligned} 0 &= a^2\dot{x} + (b^2 - a^2)yz + (a^2 - c^2)yz + (1 - a^2 - b^2)vw - (1 - c^2 - a^2)vw \\ &= a^2\dot{x} + (b^2 - c^2)(yz - vw), \end{aligned}$$

and

$$\begin{aligned} 0 &= (1 - a^2)\dot{u} - (b^2 - a^2)zv - (a^2 - c^2)yw - (1 - a^2 - b^2)yw \\ &\qquad\qquad + (1 - c^2 - a^2)zv \\ &= (1 - a^2)\dot{u} + (1 - c^2 - b^2)(zv - yw). \end{aligned}$$

These equations take the place of Euler's.

From these we get, in the first place,

$$a^2x\dot{x} + b^2y\dot{y} + c^2z\dot{z} = (b^2 - c^2)\,xvw + (c^2 - a^2)\,ywu + (a^2 - b^2)\,zuv,$$

$$\begin{aligned}(1 - a^2) . u\dot{u} &+ (1 - b^2) . v\dot{v} + (1 - c^2) . w\dot{w} \\ = \quad &(-1 + b^2 + c^2)\,(zuv - ywu) \\ &+ (-1 + c^2 + a^2)\,(xvw - zuv) \\ &+ (-1 + a^2 + b^2)\,(ywu - xvw) \\ = (c^2 - b^2)\,xvw &+ (a^2 - c^2)\,ywu + (b^2 - a^2)\,zuv;\end{aligned}$$

$$\therefore\ a^2x\dot{x} + b^2y\dot{y} + c^2z\dot{z} + (1 - a^2) . u\dot{u} + (1 - b^2) . v\dot{v} + (1 - c^2) . w\dot{w} = 0,$$

whence $a^2x^2 + \text{etc.} + (1 - a^2) . u^2 + \text{etc.} = 2T$, the equation of energy.

*XLII.

[FURTHER NOTE ON BIQUATERNIONS.]

I.

THE two expressions "twice three are six" and "six is the product of two and three," represent two different views of multiplication, although they are written down by the same shorthand formula

$$2 \times 3 = 6.$$

Accordingly they give two different interpretations of this formula.

In the first interpretation 3 is a concrete number of things, say three marbles, while 2 is not a number but an operation, namely the operation of doubling; and we may read the equation "doubling three marbles makes six marbles."

The second interpretation regards 2 and 3 as abstract numbers, and affirms the existence of a third number 6 having a definite relation to them which it is convenient to study, this third number so related being called their product; and various meanings given to the numbers 2 and 3 may lead to various concrete interpretations of the formula. Each of these views of multiplication may be extended to other things besides numbers; I propose at present to consider certain extensions of the first view.

In this we have regarded 2 as a symbol of operation, 3 as a concrete number, and 6 as a concrete number. But we may also regard all three symbols as symbols of operation, and so read the formula "doubling the triple of anything makes the sextuple of it." Such an equation as $abc = d$ will then always have two meanings:—

1. a times b times c things makes d things;

2. a times b times c times anything makes d times that thing.

That is to say, we may regard the *last* symbol in each term of the equation as either a concrete number or a symbol of operation; but all the others must be regarded as symbols of operation.

To extend this from concrete numbers to *steps* of addition or subtraction is not difficult; but it requires us to give a double meaning to the signs + and −, as well as to all numerical symbols. The first meaning is to indicate the direction of the step; thus + 3 means a step of 3 *forward*, i.e. an addition, and − 3 means a step of 3 *backward*, i.e. a subtraction. But when these symbols are attached to an operation performed upon steps, they mean *retaining* and *reversing* respectively. Thus the equation

$$(-2)(+3) = -6$$

has two meanings:—

1. Doubling a step of 3 forward and reversing it makes a step of 6 backward.

2. To triple a step and retain its direction, then to double and reverse it, is the same as to sextuple and reverse it.

These steps of addition and subtraction may be regarded as changes of position, or *vectors*, on a straight line, along which all numbers are supposed to be ranged; and by exchanging numbers for continuous quantities we may deal in this way with all vectors in a straight line. In every equation we may regard the *last* symbol in every term as either a vector or an operation; but all the others must be regarded as operations.

Assuming the law of addition of vectors in a plane, $AB + BC = AC$, we find at once the interpretation of so-called *imaginary* or *impossible* quantities in the operators which convert one vector into another. Thus [Fig. 53] if I operating on

a vector turns it counter-clockwise through a right angle, so that $I \,.\, OA = OA'$, and if

$$a = \frac{OM}{OA},\ b = \frac{MB}{OA'},$$

a and b being ratios of vectors in a line as just previously defined, then

$$OB = OM + MB = a \,.\, OA + b \,.\, OA' = (a + bI)\, OA,$$

and it is clear that $I^2 = -1$. Thus every expression of the form $a + bI$ is the ratio of two vectors.

Every vector in the plane may be represented by $a \,.\, OA + b \,.\, OA'$ by giving to a and b proper values. For shortness we may write $OA = j$, $OA' = k$; then $Ij = k$, $Ik = -j$, and we shall have

$$(a + bI)(cj + dk) = (ac - bd)\, j + (ad + bc)\, k.$$

In this way we may have to consider two classes of expressions, those in which the last symbol in every product is a vector, and all the others are ratios of vectors; and those in which all the symbols represent ratios of vectors. But, observing that

$$cj + dk = (c + dI)\, j;$$

we may make the useful convention that j is, if convenient, to be understood as written after every term; so that the complex symbol $c + dI$ will now mean *either* a ratio of two vectors as above, *or* the vector $cj + dk$. And then every expression will have a double meaning as before, and we shall have only one kind to deal with.

This artifice amounts to taking a definite vector as the unit, and representing all others by means of their ratios to the unit. The success of the artifice depends on the fact that the product of two such ratios is another ratio of the same kind.

Passing now to vectors in space, we shall find again that the operation which makes one into another is of the form $a + bQ$, where Q turns through a right angle *in the plane of the two vectors*. It will not, therefore, operate on any vector out of that plane; and the variety of these operators Q is the same as the variety of planes, or is doubly infinite. We may represent Q

as a sort of handle or axis of unit length perpendicular to the plane; and the compound operation bQ, which turns through a right angle *and* increases in the ratio 1 : b, may be represented by an axis of length b. This being so, let Q, R be two such compound operations; there is *one* vector α on which they will both operate, namely, the intersection of their planes. It is found that $Q\alpha + R\alpha$ is a vector at right angles to α, and that if S is the rectangular versor which converts α into $Q\alpha + R\alpha$, so that $S\alpha = Q\alpha + R\alpha$, then the axis of S is got by adding the axes of Q and R as if they were vectors. So we write $S = Q + R$, and we have the equation

$$(Q + R)\,\alpha = Q\alpha + R\alpha.$$

By the law of formation of $Q + R$ it is clear that

$$P + (Q + R) = (P + Q) + R = (P + R) + Q = \text{etc.},$$

this being a rectangular versor whose axis is the vector-sum of the axes of P, Q, R. Accordingly it is called $P + Q + R$; although the equation

$$(P + Q + R)\,\alpha = P\alpha + Q\alpha + R\alpha$$

does not admit of interpretation in general, because there is no vector α which is capable of being operated on by P, Q, and R.

Thus every rectangular versor may be represented by the form $xI + yJ + zK$, where IJK are the three rectangular versors whose axes are the unit-vectors ijk. And we have two kinds of complex quantities to consider; *vectors*, of the form $\rho = ai + bj + ck$, and *quaternions*, of the form $q = w + xI + yJ + zK$. The product of any number of quaternions is itself a quaternion, the units IJK being multiplied by the rules

$$IJ = K = -JI,\ \ KI = J = -IK,\ \ JK = I = -KJ,$$
$$I^2 = J^2 = K^2 = -1.$$

But we cannot multiply a vector by a quaternion in general; $q\rho$ will only have a meaning if ρ is perpendicular to the axis of q, or, which is the same thing, if $ax + by + cz = 0$. And even then, although we have the formulæ

$$Ij = k = -Ji,\ Ki = j = -Ik,\ Jk = i = -Kj,$$

we cannot find the value of $q\rho$ by direct multiplication, for the symbols Ii, Jj, Kk are unmeaning. If, however, we assume that they have the *same* value, the result of direct multiplication will come out right whenever $q\rho$ is interpretable.

The artifice, by which in the geometry of two dimensions the two kinds of complex quantities were reduced to one, is not applicable here. For although we may write

$$ai + bj + ck = (a + bK - cJ)\,i,$$

and so represent every vector by its ratio to the unit i, yet it no longer remains true that the product of two such ratios is another ratio of the same sort. But we may attain the desired reduction by a simpler method; viz., by using the symbols ijk in a double sense, as vectors and as versors. Thus in the product $q\rho$, if ρ be regarded as a rectangular versor, the true value may be obtained by direct multiplication

$$(w + xi + yj + zk)\,(ai + bj + ck),$$

the ijk now standing for what was denoted by IJK. *In certain cases*, namely, when $ax + by + cz = 0$, the expression may have another meaning, and $ai + bj + ck$ may be regarded as a vector. Here the first i being a versor and the second a vector, the assumption $ii = -1$ is unmeaning, but it is without effect on the result. Similarly an expression $pq\rho$ is *always* interpretable if all the symbols are regarded as operations on vectors; it is *sometimes* interpretable when ρ is regarded as a vector, namely, when it is perpendicular to the axis of (pq), provided we make the formal assumption that $i^2 = j^2 = k^2 = -1$ just as in the other interpretation.

Observe that the artifice by which one symbol is made to do duty for two meanings is the same in quaternions, which deal with three dimensions, and in scalars, which deal with one dimension. Namely, the signs + and −, which are originally unit-*vectors*, indicating the direction of a step forward or backward, receive the additional meaning of *versors*, retaining or reversing the direction of a vector; just as the symbols ijk mean vectors originally, and afterwards are made to mean versors too.

But in complex numbers, which deal with two dimensions, the artifice is essentially different; and that which, by a convenient inaccuracy, may be called the *product of two vectors*, has very different geometrical relations to its components. It will be shewn further on that all geometric algebras dealing with an *odd* number of dimensions resemble scalars and quaternions in this respect; while those dealing with an *even* number of dimensions resemble complex numbers.

It is clear that the versors IJK may be represented on great circles of a sphere whose centre is the origin; and if these be regarded as steps on the surface of the sphere, it will be found that the consideration of their ratios leads to the whole theory of quaternions. In this interpretation *vectors* can only be represented by points on the sphere supposed to have definite *weights* attached to them, proportional to the length of the corresponding vectors. Here then we have the geometric algebra of three dimensions interpreted by means of a space of two dimensions which has constant positive curvature, namely the surface of a sphere.

In the same way we may interpret the algebra of a space of four dimensions, which cannot be imaged, by means of a space of three dimensions having constant positive curvature, of which a clear mental picture may be formed.

Consider any vertical line, and a series of horizontal planes cutting it at right angles. In ordinary or Euclidian geometry these planes intersect on the *horizon*, which is a straight line infinitely distant. In the geometry of a space of constant positive curvature, or *elliptic* geometry, the horizon is at a certain finite distance in all directions from the vertical line with which we started; it belongs to that particular line, which is called its *polar*, and is not the same for all vertical lines. Although it appears to be a great circle when viewed from the neighbourhood of its polar, yet if we were to go to it and examine it we should find it straight. Points of it which are in opposite directions from a point on the polar are really identical; and every straight line in this space resembles a circle in being of finite length, so that if we travel far enough along it we shall arrive at our starting point. Every straight line

has a polar line, which is the intersection of all planes at right angles to it.

Let us take a very small circle on a sphere, and suppose it to expand, keeping always the same centre. At the beginning the circle will be concave inside and convex outside; but when the expansion has gone on far enough it will become a great circle of the sphere, which is of the same shape on both sides, or is *straight* so far as the surface of the sphere is concerned. So if in Euclidian space we take a sphere and suppose it to expand, keeping always the same centre, it will continue to be concave inside and convex outside so long as it is finite; but when the radius has become infinite, the inside in one direction is the same as the outside in the opposite direction, opposite points being identical; thus the sphere is of the same shape on both sides, or is a *plane*, viz., the plane at infinity. In elliptic space, just as in geometry on the surface of a sphere, this takes place for a *finite* length of the radius, not for an infinite length; for every point there is a sphere having its centre at that point, which is also a plane. Or, which is the same thing, every point has a polar plane which is the locus of all points situate at a certain distance from it; this distance is called a *quadrant*. So also every plane has a certain point, called its *pole*, which is distant a quadrant from every point in the plane. All lines and planes perpendicular to the plane pass through its pole, and conversely. The polar lines of all lines in the plane pass through its pole, and so do the polar planes of all points in the plane.

When two lines are polars of one another, every point of one is distant a quadrant from every point of the other; hence the polar planes of all points on one pass through the other. Every line which is at right angles to one meets the other, and conversely.

In the *Proceedings of the London Mathematical Society*, [Vol. IV. p. 381—395*] I have given a sketch of a geometric algebra adapted to this elliptic geometry of space, which I have there called Biquaternions.

In the interpretation of quaternions on the surface of a sphere, rectangular versors are represented by quadrants of

* [XX. supra.]

great circles. We may represent such a versor accompanied by a tensor, for example xi, by an arc AB measured on the great circle i, so that $\tan AB = x$. This being so, AB differs from a vector in a plane in a most important way; for while a vector in a plane is unaltered by being moved parallel to itself in any direction, AB can only be slid along its great circle, and must not be moved out of it. We shall have to consider similar quantities in the elliptic geometry of three dimensions; namely quantities represented by a length marked off on a certain straight line, which are unaltered when the length is slid along the line but not in any other case. They are as it were *vectors having position.* A vector represents the translation-velocity of a rigid body, which is everywhere the same; these quantities will represent the *rotation-velocity* of a rigid body, which is about a certain definite axis. For this reason I have called them *rotors* (short for *rotators*) by analogy with Hamilton's word *vector.* They are added together according to the law of composition of forces and rotations. That is to say, if a rotor P [fig. 54] along OA added to a rotor Q along OB gives a rotor R along OC making angles α, β with them, then OC is in the plane of OA, OB, and

$$P : Q : R = \sin\beta : \sin\alpha : \sin(\alpha+\beta);$$

this determines the position and magnitude of the resultant. We cannot use the parallelogram construction as in the addition of vectors, for the (elliptic) geometry of the plane AOB is the same as that of the surface of a sphere when opposite points are regarded as identical, and no parallelogram can be drawn on it.

Since any two great circles of a sphere meet one another (in *one* point, according to our present convention) it follows that any two rotors have a single rotor which is their resultant or sum. But in three dimensions this is not the case; if the axes of two rotors do not meet their sum is not equal to any one rotor. We may however find two other rotors which have the same sum, and that in an infinite number of ways. Of these ways one is of the greatest importance, namely, that in which the axes of the two rotors are polars of one another. If

we regard each rotor as representing rotation about its axis, each of these rotations is equivalent to a translation along the other axis. Thus rotation about a vertical line is translation along the horizon, and *vice versa.* Hence the resultant of the two rotations may be regarded as a screw motion about either of the axes. As this describes the most general motion of a rigid body, I have proposed to call the quantity which represents it a *motor.* We shall say then that the sum of two rotors which do not meet is a motor, and that every motor has two axes which are polars of one another.

This being so, let three planes at right angles be drawn through a point O, and let unit rotors along their intersections be denoted by ijk. Then any rotor through the point O is denoted by $ai+bj+ck$. The ratio of two such rotors is a quantity of the form $w+xi+yj+zk$, if we let ijk mean also rectangular versors whose axes are the rotors ijk. In fact we are merely applying the results of quaternions to vectors passing through a fixed point and their ratios.

Let now ω be the operation which converts any rotor into an equal rotor along the polar line of its axis. Then $\omega i, \omega j, \omega k$ [fig. 55] will be rotors along the lines of intersection of the polar plane of O with the three rectangular planes through O. And since (as it is easy to see) any rotor may be resolved into two, one passing through O, and the other lying in the plane PQR, whereof the former is compounded of ijk, and the latter of $\omega i, \omega j, \omega k$, it follows that the expression for the sum of any number of rotors (i.e. for a motor) is of the form

$$ai+bj+ck+\omega(fi+gj+hk) = \alpha+\omega\beta, \text{ say.}$$

Suppose now that the *versors* ijk are allowed to operate, not only on rotors through O which meet their axes, but on any rotors which meet them at right angles. Then we shall have

$$i(\omega j) = \omega k = \omega ij\,;\; j(\omega k) = \omega i = \omega jk\,;\; k(\omega i) = \omega j = \omega ki,$$

which equations shew that ω is *commutative* with the symbols i, j, k.

Moreover, if ω turns a rectangular versor into a similar versor about the polar line of its axis, we shall have the equations

$$\omega i \,.\, j = \omega k,\ \omega j \,.\, k = \omega i,\ \omega k \,.\, i = \omega j,$$

which shew that the two meanings thus attributed to ω lead to no contradiction. Lastly, we have

$$\omega i \,.\, \omega j = k,\ \ \omega k \,.\, \omega i = j,\ \ \omega j \,.\, \omega k = i$$

from which we get $\omega^2 = 1$.

From this equation we may draw a very important consequence. Writing

$$\xi = \tfrac{1}{2}(1 + \omega), \quad \eta = \tfrac{1}{2}(1 - \omega),$$

and therefore

$$1 = \xi + \eta, \quad \omega = \xi - \eta,$$

we find

$$\xi^2 = \tfrac{1}{4}(1 + 2\omega + \omega^2) = \xi\,; \ \ \eta^2 = \tfrac{1}{4}(1 + 2\omega + \omega^2) = \eta,$$
$$\xi\eta = \tfrac{1}{4}(1 - \omega^2) = 0.$$

Hence every motor $\alpha + \omega\beta$ may be written in the form

$$(\xi + \eta)\,\alpha + (\xi - \eta)\,\beta \text{ or } \xi\,(\alpha + \beta) + \eta\,(\alpha - \beta).$$

Consider two motors

$$\xi\alpha + \eta\beta,\ \xi\gamma + \eta\delta, \text{ and let } \gamma\alpha^{-1} = p,\ \delta\beta^{-1} = q,$$

so that p and q are known quaternions; then we have

$$(\xi p + \eta q)\,(\xi\alpha + \eta\beta) = \xi p\alpha + \eta q\beta$$
$$= \xi\gamma + \eta\delta.$$

Thus the ratio of two motors is a quantity of the form $\xi p + \eta q$, or, which is the same thing,

$$s + \omega t \ (\text{if } 2s = p + q,\ t = 2p - q),$$

where p, q, s, t, are quaternions. This combination of two quaternions I have called a *Biquaternion**.

* [I am indebted to Dr Spottiswoode for the title I have given to this paper. It is designated as ǐ by Prof. Clifford, but I have not come across any other papers of the series.]

NOTES ON BIQUATERNIONS.

1. AXES of motor $\alpha + \omega\beta$, or $\xi\gamma + \eta\delta$.

If μ be an axis of a motor, the motor is a multiple of the rotor μ together with a multiple of the polar rotor $\omega\mu$. Thus we may write

$$\xi\gamma + \eta\delta = (h + \omega k)\,\mu.$$

Operating by ξ, η, we have

$$\xi\gamma = (h + k)\,\xi\mu,$$
$$\eta\delta = (h - k)\,\eta\mu,$$

whence

$$\mu = \frac{\xi\gamma}{h + k} + \frac{\eta\delta}{h - k}.$$

Thus we have an indeterminate representation of any motor as the sum of two polar motors. But if μ is a *rotor*, we have

$$\frac{T\gamma}{h + k} = \frac{T\delta}{h - k},$$

whence

$$\frac{h}{k} = \frac{T\gamma + T\delta}{T\gamma - T\delta},$$

so that

$$\mu = \xi \,.\, U\gamma + \eta \,.\, U\delta,$$

and

$$\omega\mu = \xi \,.\, U\gamma - \eta \,.\, U\delta.$$

But the component rotors along the axes are

$$(T\gamma \pm T\delta)\,(\xi \,.\, U\gamma \pm \eta \,.\, U\delta).$$

For the sum of these is easily seen to be equal to the original motor $\xi\gamma + \eta\delta$.

* [These "Notes" were apparently written when Prof. Clifford was in the enjoyment of vigorous health, and occupy three pages of MS. pinned together. They are, I should say, intended to be supplementary to his *London Mathematical Society* paper, XX. supra.]

2. Sum of motors of different pitches whose axes meet at right angles.

Let the motors be $(a+\omega b)\,j$ and $(c+\omega d)\,k$; the sum is of course

$$aj+ck+\omega\,(bj+dk)$$

or

$$\tfrac{1}{2}\,\xi\,(\overline{a+b}\,.\,j+\overline{c+d}\,.\,k)+\tfrac{1}{2}\,\eta\,(\overline{a-b}\,.\,j+\overline{c-d}\,.\,k).$$

Thus an axis of the sum is

$$\xi\,\frac{\overline{a+b}\,.\,j+\overline{c+d}\,.\,k}{\sqrt{(a+b^2+c+d^2)}}+\eta\,\frac{\overline{a-b}\,.\,j+\overline{c-d}\,.\,k}{\sqrt{(a-b^2+c-d^2)}}.$$

Now this is a rotor at right angles to i and at an angular distance θ from the origin such that

$$\cos 2\theta=\frac{a^2-b^2+c^2-d^2}{\sqrt{(a+b^2+c+d^2)\,(a-b^2+c-d^2)}}$$

$$\tan 2\theta=\frac{2\,(bc-ad)}{a^2-b^2+c^2-d^2}.$$

*XLIII.

ON THE CLASSIFICATION OF GEOMETRIC ALGEBRAS*.

1806 Argand, *Manière de représenter les quantités imaginaires.*
Buée, *Mém. sur les qu. imag.*
1827 Möbius, *Barycentrischer Calcul.*
1831 Gauss.
1834 Peacock, *Doctrine of Operations in Algebra.*
1843 Hamilton, *Quaternions.*
1844 Grassmann, *Lineale Ausdehnungslehre.*
1845 Saint-Venant, *Multiplication of vectors.*
1848 Kirkman, *Pluquaternions and Homoid Products.*
1853 Cauchy, *Clefs Algébriques.*
1862 Grassmann, *Ausdehnungslehre.*
1870 Peirce, *Linear Associative Algebra.*

In the Barycentric Calculus a point is represented by a complex number which is a linear syzygy of symbols each representing a fixed point; the coefficients are coordinates. By regarding ab the ordinary symbol for a line joining two points, as of the nature of a product, and so *distributive*, we arrive at Grassmann's extensive quantities. For then we must put $aa=0$, i.e. $(a_1\iota_1+a_2\iota_2)^2=0$, which requires $\iota_1\iota_2=-\iota_2\iota_1$, and then $ab=-ba$ always. If there are n independent units, we may consider in such an algebra scalars or quantities of order 0, n of order 1, $\tfrac{1}{2}n\,.\,(n-1)$ of order 2, etc., 1 of order n; in all 2^n *vids*, to borrow Peirce's term. Every intelligible expression is however homogeneous; if a product of points, the coefficients are determinants.

In the theory of Quaternions the symbols ijk, when used as multipliers, represent not things but operations of turning; thus $i^2=-\dot{1}$ and not 0. But regarding them as vectors, we may use them to represent the geometry of the plane passing through the ends of ijk, on the principles of the barycentric calculus. Thus $\rho=ix+jy+kz$ will represent the point where it cuts the plane, with a *weight* $x+y+z$. The *Grassmann* algebra will be reproduced if we attend only to the *vector* part of binary products, and the *scalar* part of ternary. Physical considerations however lead us to regard i^2 as a scalar (not zero) even

* [The "forewords" are the abstract which Prof. Clifford communicated to the *London Mathematical Society*, on March 10th, 1876 (*Proceedings*, Vol. VII. p. 135). The paper, which is unfinished, was found amongst his MSS.]

when i is regarded as a vector (not a versor). For these purposes it does not matter whether it is put $= -1$ or $+1$.

I propose here to extend this assumption to the Grassmann representation in general: i.e., I take n units $\iota_1 \iota_2 \ldots \iota_n$ such that $\iota_r^2 = +1$, and $\iota_r \iota_s = -\iota_s \iota_r$. All products of linear factors must therefore contain either terms of odd order only, or terms of even order only. There can be no term of higher order than the n^{th}, and the whole number of terms is 2^n as with Grassmann; i.e. in both cases we have a linear associative algebra of 2^n units, but the Grassmann algebra is *nilpotent*, and only homogeneous forms occur; while this is *idempotent*, and admits of *odd* forms and *even* forms, which are not in general homogeneous. It is convenient to use *selective* symbols $V_0, V_1 \ldots V_n$ analogous to Hamilton's S and V, for picking out those parts of any expression which are of order $0, 1 \ldots n$ respectively.

The Quaternion symbols satisfy the equation $ijk = -1$, and this together with the assumption $\rho^2 =$ scalar gives all the laws of their multiplication. If we put $\omega = \iota_1 \iota_2 \ldots \iota_n$, this means that in the case $n = 3$ we may take ω to be a scalar. There is here a very important distinction between the cases n odd and n even; in the former case ω is commutative with the symbols ι, or $\omega\iota = \iota\omega$, in the latter case $\omega\iota = -\iota\omega$. Hence when n is odd ω acts as a scalar, when n is even it acts as a vector. In putting $\omega =$ to a scalar in the former case we are conveniently representing two different things by the same symbols, because they have the same laws of combination. We thus reduce the algebra to 2^{n-1} units when n is odd. When n is even the symbol ω is of even order. In all cases we may if we like consider separately the even units and the odd units; the former make an algebra by themselves, and by restricting ourselves to these we get the same result as by putting $\omega = \pm 1$. Thus, $n = 3$, the even algebra gives quaternions at once. When n is even we may still further simplify; for the symbol ω belongs to the even algebra, and this consists of 2^{n-2} terms (i.e. the even terms up to order $n - 2$ and half of these) together with the products of these terms by ω, which is commutative with them. Thus $n = 4$, the even algebra gives biquaternions, and the general expression is $q + \omega r$ where q, r are quaternions. For n odd, we may represent the odd algebra by the even algebra; this amounts to making $\omega = \pm 1$.

THE extensive quantities of Grassmann, which Hankel has called alternate numbers, namely symbols which have the property of *polar* multiplication $ab = -ba$ and whose square vanishes, $a^2 = 0$, serve to conveniently represent the projective geometry of n dimensions. In plane geometry, for example, let the symbols ι_1, ι_2, ι_3 represent three points; then

$$a = a_1 \iota_1 + a_2 \iota_2 + a_3 \iota_3$$

will represent a point which is the centre of inertia of masses a_1, a_2, a_3 placed at the fundamental points respectively. If the

products $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$ be taken to mean the lines joining the fundamental points, then the product

$$\begin{aligned} ab &= (a_1\iota_1 + a_2\iota_2 + a_3\iota_3)(b_1\iota_1 + b_2\iota_2 + b_3\iota_3) \\ &= (a_2b_3 - a_3b_2)\,\iota_2\iota_3 + (a_3b_1 - a_1b_3)\,\iota_3\iota_1 + (a_1b_2 - a_2b_1)\,\iota_1\iota_2 \end{aligned}$$

represents the line joining the points a, b; for the coefficients of the binary products $\iota_2\iota_3, \ldots$ are clearly the coefficients in the equation of that line referred to the fundamental triangle.

In like manner the ternary product

$$abc = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \iota_1\iota_2\iota_3$$

is proportional to the area of the triangle abc, and vanishes when the three points are collinear. And so on.

The system of quaternions differs from this, first in that the squares of the units, instead of being zero, are made equal to -1; and secondly in that the ternary product $\iota_1\iota_2\iota_3$ is made equal to -1. The interpretation is at the same time extended to three dimensions, but with this restriction: that whereas the alternate units represent *any* three points in a plane, and the system deals primarily with projective relations, Hamiltonian units represent three vectors at right angles, and the system is the natural language of metrical geometry and of physics.

I shall now examine the consequence of making, in a system of n alternate numbers $\iota_1, \iota_2, \ldots \iota_n$, the *first* of the modifications just named; namely I shall suppose that the square of each of the units is -1. I shall then enquire whether any assumption can be made which is analogous to the second Hamiltonian law, $\iota_1\iota_2\iota_3 = -1$.

In the system of Grassmann every product of linear factors is homogeneous in the units; namely the product of m linear factors is a linear function of the m-ary products when m is not greater than n, and is zero when $m > n$. But if we make the square of every unit equal to -1, it is clear that the product of m factors (m not greater than n) may contain terms of the order $m, m-2, m-4, \dots m-2r, \dots$, since the order is reduced by 2 for every substitution of -1 for ι^2. Consequently every such product consists of terms which are either all of odd order or all of even order. The same thing is true when m is greater than n, except that the highest order which can occur is either n or $n-1$, according as $m-n$ is even or odd. The products of an even number of the units will therefore form an algebra complete in itself. We have altogether one term of order 0, n of order 1, $\frac{1}{2}n(n-1)$ of order 2, … one of order n; that is to say $1+n+\frac{1}{2}n(n-1)\dots+n+1=2^n$ terms. Moreover we have

$$\text{terms of even order} - \text{terms of odd order}$$
$$= 1 - n + \tfrac{1}{2}n(n-1) - \dots + (-1)^n = (1-1)^n = 0,$$

or the number of terms in the even algebra is equal to the number of terms in the odd algebra, namely 2^{n-1}.

Let us now write ω for the product of all the units, $\omega = \iota_1\iota_2 \dots \iota_n$, and enquire into the value of ω^2. This is

$$\omega^2 = \iota_1\iota_2 \dots \iota_n\iota_1\iota_2 \dots \iota_n,$$

which after p interchanges of contiguous symbols can be transformed into

$$\omega^2 = (-)^p\, \iota_1^2\iota_2^2 \dots \iota_n^2 = (-)^{p+n}.$$

To find the value of p, we observe that it takes $n-1$ changes to bring the second ι_1 between the first ι_1 and ι_2, then $n-2$ changes to bring the second ι_2 between the first ι_2 and ι_3, and so on. Therefore $p+n=1+2+\dots+n=\frac{1}{2}n(n+1)$; and it follows that $\omega^2=\pm 1$ according as $\frac{1}{2}n(n+1)$ is even or odd, or according as n belongs to the forms $4m$, $4m+3$ or $4m+1$, $4m+2$.

Next let us observe that $\iota_1\omega = -\iota_2\iota_3 \dots \iota_n$, but $\omega\iota_n = (-)^n\, \iota_2\iota_3 \dots \iota_n$; that is to say, $\iota_1\omega = \pm\, \omega\iota_1$ according as n is odd or even. Thus the multiplication of ω with the units is polar or commutative according as n is even or odd.

We are thus led to distinguish four classes of geometric algebra, which are characterized by the sign of ω^2 and the nature of the multiplication $\omega\iota$.

Class I. $n \equiv 0$ (mod. 4). $\omega^2 = 1$, $\omega\iota = -\iota\omega$. The even algebra contains the symbol ω, which is commutative with all even terms. Since every term of order $r > \frac{1}{2}n$ can be expressed as the product of ω into a term of order $n-r$, while the terms of order $\frac{1}{2}n$ divide themselves into complementary pairs, each of which is ω multiplied by the other, this algebra contains 2^{n-2} symbols, together with their products by ω. Thus for $n=4$, the most general expression in the even algebra is

$$w + x\iota_2\iota_3 + y\iota_3\iota_1 + z\iota_1\iota_2 + \omega\,(w' + x'\iota_2\iota_3 + y'\iota_3\iota_1 + z'\iota_1\iota_2) = q + \omega r,$$

which is precisely what I have elsewhere called a *biquaternion*, because the products $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$, satisfy the laws of the Hamiltonian symbols i, j, k, and therefore the quantities q, r are quaternions in the ordinary sense.

The most general expression in the odd algebra consists also of eight terms, namely it is

$$x\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4 - \omega\,(x'\iota_1 + y'\iota_2 + z'\iota_3 + w\iota_4).$$

The coefficients are purposely so arranged as to bring out the fact that this expression may be derived from the preceding by multiplying it by $\iota_4\omega$. It does not, however, follow that we may always represent the vector $x\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4$ by the proportional quantity of the even algebra

$$x\iota_2\iota_3 + y\iota_3\iota_1 + z\iota_1\iota_2 + w'\omega.$$

For if α, β be any two vectors, the quantity $\iota_4\omega\alpha\,.\,\iota_4\omega\beta$ is not in general proportional to $\alpha\beta$. This would require that the multiplication of α or β with $\iota_4\omega$ should be either polar or commutative; whereas in general it is neither.

Class II. $n \equiv 1$ (mod. 4) $\omega^2 = -1$, $\omega\iota = \iota\omega$. The odd and even algebras may be included in the same formulæ by putting ω equal to the scalar $\sqrt{-1}$; the case is thus closely analogous to that of $n \equiv 3$ (mod. 4).

Class III. $n \equiv 2$ (mod. 4), $\omega^2 = -1$, $\omega\iota = -\iota\omega$. Here ω has clearly the properties of a unit vector, and the system may always be treated as a degenerate case of the next. The general expression of the even algebra may be got from the general expression of the odd algebra by multiplying it by $\omega\iota_n$.

*XLIV.

ON THE THEORY OF SCREWS IN A SPACE OF CONSTANT POSITIVE CURVATURE.

Polar lines.

THE word *line* will be here used to mean *great circle* except where ambiguity might occur. To every line in a space of constant positive curvature corresponds a polar line such that every point on it is distant a quadrant from every point on the line. This is in fact its polar line in regard to the *absolute*, a quadric whose equation referred to a quadrantal tetrahedron is

$$x^2 + y^2 + z^2 + w^2 = 0.$$

Translation = rotation about polar (Klein).

The rotation of a rigid body about any line is a sliding of the body along the polar line; and conversely a *translation* or sliding without twist along any line is a rotation about the polar line. A translation therefore in such a space is not merely a vector having magnitude and direction, but a quantity having also a definite position.

Axes of Screw.

The most general motion of a rigid body is made up of translation along a certain axis and rotation about it; or we may say that it is made of rotations about two polar lines. These factors in the motion may be combined in any order without affecting the result. Let α and α_1 be two polar rect-

angular versors (rotations through a right angle about two polar lines), then the product

$$\alpha^p \alpha_1^q$$

is the most general representation of the motion of a rigid body. This may be regarded either as a twist $p\frac{\pi}{2}$ about a screw whose axis is α and pitch $\frac{q}{p}$, or as a twist $\frac{q\pi}{2}$ about a screw whose axis is α_1 and pitch $\frac{p}{q}$. Thus a screw has two axes which are polars of each other, and its pitch is only completely defined when we have picked out one of these axes for attention.

Rotor Sum.

A *rotor* has magnitude, direction, and position. The rotor AB will equal the rotor CD when they are in the same line, of equal length, and of like sense. Two rotors therefore will not in general meet; and the sum of two rotors will in general be a quantity of the nature of a twist or a wrench, which I call a *motor*. If we suppose a rigid body to have rotation velocities about these rotors proportional to their lengths, the resultant of these will be a twist velocity about the screw which is their sum.

Rotor Ratio = Twist × Tensor.

The *ratio* of two rotors is a twist multiplied by a tensor. For there are two polar lines which cut the two rotors at right angles; hence one vector can be made into the other by rotations about these lines combined with an alteration of length.

Scale of quantities.

We have thus a certain scale of quantities, each of which is obtained by connecting the notion of magnitude with a certain geometrical form. The *ratios* of these quantities are the subject-matter of successive algebras.

Geometrical Form.	Quantity.	Examples.	Ratio.
Sense on straight line	Vector on straight line		Real quantity (±) of algebra
Direction in plane	Vector in plane		Complex ratio $a+bi$
Direction in space	Vector in space	Couple; transl. vel. of rigid body	Quaternion
Axis	Rotor	Force; rotation velocity	Twist
Screw	Motor	System of forces; twist velocity	Biquaternion

It is to be observed that this is not an uniformly ascending series, but that the theoretic order has been broken to meet the requirements of practical application. After Hamilton's quaternion should come the ratio of two vectors in four dimensions; now a *rotor* may be regarded as the *logarithm* of such a ratio *when the vectors are at right angles.*

Ratio of polar rotors.

Let α and β be two rotors; it is clear that the same twist which converts α into β converts also the polar of α into the polar of β. Let that twist which converts any rotor into its polar be called ω; {this is a rectangular twist about any screw of pitch 1 whose axis meets the rotor perpendicularly} then this result may be written

$$\frac{\beta}{\alpha}=\frac{\omega\beta}{\omega\alpha}\equiv q, \text{ suppose.}$$

Polar motor.

The sum of any number of rotors is a *motor;* and it is known that every motor may be expressed as the sum of two polar rotors, that is to say in the form

$$\alpha+k\,.\,\omega\alpha\equiv A,$$

where κ is the pitch of the motor in regard to the axis α. The motor

$$k\alpha+\omega\alpha\equiv\omega A$$

will be called the polar of A. In regard to the axis α it has a pitch reciprocal of the former one.

Every motor may be regarded as the sum of two, one having any perfectly arbitrary pitch and the other the pitch reciprocal to it. For the equation

$$\alpha+k\omega\alpha=x\,(\alpha+h\omega\alpha)+y\,(h\alpha+\omega\alpha)$$

gives

$$1=x+hy,$$
$$k=hx+y,$$

and thence

$$x=\frac{1-kh}{1-h^2},\ y=\frac{k-h}{1-h^2}.$$

Ratio of motors.

If two motors have the same pitch their ratio is a twist. For we have seen above that

$$\beta=q\alpha,\qquad \omega\beta=q\omega\alpha,$$

from which it follows that

$$\beta+k\omega\beta=q\,(\alpha+k\omega\alpha).$$

Let the ratio of $\omega\beta$ to α be called the *polar* twist of q; so that we shall write

$$\omega q=\frac{\omega\beta}{\alpha}=\frac{\beta}{\omega\alpha}.$$

{Observe that q is a *finite* twist, and not a *motor*, or *twist velocity;* so that this must not be confounded with the previous meanings of *polar* and of ω.}

The ratio of any two motors is the sum of two polar twists. Let $\alpha+h\omega\alpha$, $\beta+k\omega\beta$ be the two motors; and let $\frac{\beta}{\alpha}=q$. Then

$$\frac{\beta+k\omega\beta}{\alpha+h\omega\alpha}=\frac{(1-kh)\,(\beta+h\omega\beta)+(k-h)\,(h\beta+\omega\beta)}{(1-h^2)\,(\alpha+h\omega\alpha)}$$
$$=\frac{1-kh}{1-h^2}\cdot q+\frac{k-h}{1-h^2}\cdot\omega q.$$

XLV.

REMARKS ON A THEORY OF THE EXPONENTIAL FUNCTION DERIVED FROM THE EQUATION $\frac{du}{dt}=pu$*.

After shewing that a quantity which is equally multiplied in equal times must always increase at a rate proportional to itself, or which is the same thing, satisfy the equation $\frac{du}{dt}=pu$, we may call the ratio p of the growth to the growing quantity, the *intrinsic rate*. If we then define as follows:—$\epsilon^{px}.\,u$ is the result of making u grow at the intrinsic rate p for x seconds. This applies so far to pure quantity only; but if we regard u as

(*a*) a vector on a line, i.e., a signed magnitude,

(*b*) a vector in a plane, i.e., a complex magnitude, or

(*c*) a vector in space,

then p being the ratio of two such vectors is respectively

(*a*) a signed ratio; justifying ϵ^{-px},

(*b*) a complex ratio; from which it becomes obvious that $\epsilon^{i\theta}=\cos\theta+i\sin\theta$,

(*c*) a quaternion; from which we get Hamilton's theory of exponential functions of quaternions.

A semijustification of the symbolic form of Taylor's Theorem (independently of the series) is also obtained by this method.

The *series* for $\epsilon^{px}.\,u$ in all cases where it holds good may then be either established by Taylor's theorem, or shewn to be a solution and the only one of the equation $\frac{du}{dt}=pu$ between values 0 and x of t.

* [From *Proceedings of London Mathematical Society*, Vol. IV. No. 47, p. 111.]

XLVI.

NOTES ON VORTEX-MOTION, ON THE TRIPLE GENERATION OF THREE-BAR CURVES, AND ON THE MASS-CENTRE OF AN OCTAHEDRON*.

(i) On Vortex-Motion.

Let σ be the velocity, and ω the rotation, at any point of a moving substance. It is known that $2\omega=V\nabla\sigma$; viz., this is equivalent to the three equations ordinarily written thus:

$$2\xi=\delta_z v-\delta_y w,$$
$$2\eta=\delta_x w-\delta_z u,$$
$$2\zeta=\delta_y u-\delta_x v.$$

If, moreover, k be the *expansion*, or the logarithmic rate of change of the volume, we have $k=-S\nabla\sigma$; viz., this is $\delta_x u+\delta_y v+\delta_z w$. Hence the quaternion q, $=-k+2\omega$, is simply $\nabla\sigma$. The problem solved by Stokes as a general question of Analysis, and subsequently by Helmholtz for the special case of fluid motion, may be stated as follows:—Given the expansion and the rotation at every point of a moving substance, it is required to find the velocity at every point. In symbols, it being known that $q=\nabla\sigma$, and q being given, it is required to find σ.

The solution of the problem is exhibited in a very simple form if we consider it a little more generally.

A quaternion q is given at every point of space; it is required to find a quaternion r so that $q=\nabla r$. The solution is that $\nabla q=\nabla^2 r$, and therefore r is the potential of ∇q; that is,

$$r_a=\int\frac{\nabla q_b\,.\,dv_b}{D_{ab}},$$

* [From the *Proceedings of the London Mathematical Society*, Vol. IX. No. 125, pp. 26—28.]

where r_a means the value of r at the point a, δv_b means an element of volume at the point b, and D_{ab} the distance between the two points a, b. If r is a pure vector, so that $Sr=0$, q must satisfy a certain condition, namely, we must have $S\nabla q=0$ everywhere. This is identically satisfied if $q=-k+2\omega$, where ω is the rotation at any point of a moving substance; for

$$2S\nabla\omega = S\nabla^2\sigma = 0.$$

(ii) On the Triple Generation of Three-bar Curves.

The theorem on which the triple generation of three-bar curves depends has been stated as follows by Prof. Cayley:— Let the triangles [Fig. 56] *deo*, *fog*, *ohk* be similar, and the figures *adof*, *cgok*, *bhoe* parallelograms. Then the triangle *abc* will be similar to *deo*, &c.

The proof of this is intuitive if we consider the operation which converts *oh* into *ok*. This operation consists in turning through the angle *hok*, and altering the length in the ratio *oh* : *ok*. The same operation converts *eb* into *gc*, *de* into *do*, and therefore into *af*, and *ad* or *fo* into *fg*. Hence *ad*, *de*, *eb* are converted by this operation into *fg*, *af*, *gc*, and therefore the whole line *ab* is converted by it into the whole line *ac*. That is to say, the triangle *abc* is similar to *ohk*, as was to be proved.

If we complete the parallelogram *adel*, this amounts to saying that the broken lines *aleb*, *afgc* are similar to one another.

If one of the three-bar systems is a crossed rhomboid, the other two are kites. This would of course follow from the known fact that the path of the moving point in both these cases is the inverse of a conic. But it is also intuitively obvious as soon as the figure is drawn, and thus supplies an elementary proof that the path *is* the inverse of a conic in the case of a kite, which is not otherwise easy to get.

(iii) On the Mass-centre of an Octahedron.

Let *af*, *bg*, *ch* [Fig. 57] be three finite lines not meeting. By an *Octahedron* I mean the solid whose eight faces are *abc*, *acg*, *agh*, *ahb*, *fbc*, *fcg*, *fgh*, *fhb*. If this solid figure be filled with matter of uniform density, its mass-centre may be found by a very simple construction.

The solid is girdled by three skew quadrilaterals *bcgh*, *cahf*, *abfg*. Now the middle points of the sides of any skew quadrilateral are in one plane. Draw, then, three planes bisecting the sides of these quadrilaterals, and let them meet in a point *k*; which, following Sylvester in a paper to be presently mentioned, I will call the *mass-centre*. Let also *m* be the mean point of the six vertices *a*, *b*, *c*, *f*, *g*, *h*; it is the mass-centre of the triangle formed by the middle points of *af*, *bg*, *ch*. To find *s*, the mass-centre of the solid, join *km* and produce it to *s* so that $ms = \frac{1}{2}km$.

The proof is that the solid is the sum of the four tetrahedra *afbc*, *afcg*, *afgh*, *afhb*. Now the mass-centre of a tetrahedron is the mean point of its vertices; consequently the line joining the mass-centre of *afbc* to the middle point of *gh* is divided by the point *m* in the ratio 1 : 2. The same is true of the other three tetrahedra and the middle points of *hb*, *bc*, *cg*. Hence the mass-centres of the four tetrahedra are in one plane passing through the point *s* found by the above construction, and therefore the mass-centre of the whole solid is in this plane. So also it is in the other two planes determined by dividing the solid into tetrahedra having the common edge *bg* and the common edge *ch* respectively. Therefore it coincides with the point *s*.

This remark was suggested to me by Sylvester's construction for the mass-centre of a tetrahedral frustum, of which it is a simple extension. In fact, by making the pairs of faces *abh*, *ahg*; *acg*, *cfg*; *cbf*, *bhf* to be respectively coplanar, we pass at once to that particular case.

XLVII.

GEOMETRICAL THEOREM*.

THE following is a proof by Pure Geometry of the proposition given by Mr Ferrers, in Vol. I. p. 159, as the reciprocal of Prof. Cayley's theorem about rectangular hyperbolæ.

The other corollaries are well known propositions. This form of proof suggests the corresponding propositions in Geometry of three dimensions.

1. *If a series of conics be inscribed in the same quadrilateral, their directors will all have the same radical axis.*

The tangents drawn from any point to a series of conics inscribed in the same quadrilateral form a pencil in involution. If therefore the point be such that any two of the conics subtend right angles at it, all the conics will subtend right angles. But the director of a conic is the locus of intersection of tangents at right angles to each other. Therefore the intersections of any two directors are points on all the others. Q. E. D.

2. *The foci of the quadrilaterals formed by five lines lie on a circle.*

For they all lie on the director of the conic touching the five lines.

3. *The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.*

For the diagonals may be regarded as very thin ellipses inscribed in the quadrilateral.

* [From the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, Vol. III. pp. 31, 2, 1866.]

4. *Every conic through the intersection of two rectangular hyperbolæ is a rectangular hyperbola.*

This is the reciprocal of (1) with regard to either of the points of intersection. This is Prof. Cayley's theorem of Vol. I., p. 77.

5. *The directrix of a parabola passes through the polar centre of every circumscribed triangle.*

This follows from (1) by sending one side of the quadrilateral to infinity. For the circles on the diagonals of the quadrilateral as diameters become then the perpendiculars of the triangle.

6. *The polar centres of the triangles formed by four straight lines lie on the line joining the foci of the quadrilateral.*

This line is the directrix of the inscribed parabola, by (1).

XLVIII.

ON TRIANGULAR SYMMETRY*.

WE make the properties of a conic intuitive by studying it under the form of a circle; or those of a quadrilateral, by studying it under the form of a square. This simplification depends upon the projective property of a right angle, viz., that it divides harmonically the chord at infinity of a circle. By means of this property we interpret as general, propositions whose truth we see intuitively through the symmetry of the figure. This kind of symmetry (that of a circle or square) I call the *symmetry of the right angle*, or *rectangular symmetry*.

From the symmetry of two lines we ascend immediately to the symmetry of three lines, or of the equilateral triangle. This is exemplified in the Rhombohedral System of Crystals, just as Rectangular Symmetry is exemplified in the Pyramidal System. I want in this note to call attention to the uses of Triangular Symmetry in presenting general propositions under an intuitive form.

The projective property of an equilateral triangle is this: *it determines on the line at infinity a point-cubic whose Hessian is the circular points.* Given four lines, we may project one of them to infinity so that the other three shall form an equilateral triangle; for we have only to construct the Hessian of the point-cubic determined upon that one by the other three, and then project this Hessian into the circular points. Similar problems are

B. To project at once a conic and triangle into a circle and an equilateral triangle.

* [From *Mathematics from the Educational Times*, Vol. IV. pp. 88, 9.]

C. To project at once two conics and a triangle into two rectangular hyperbolæ and an equilateral triangle.

D. To project at once two triangles into equilateral triangles.

In each of these we have the problem of finding the line which has to be projected to infinity; this problem admits, in the three cases respectively, of 43, 76, and 108 solutions. The *triangle* might of course be replaced by a *cubic*, to be so projected that its asymptotes should form an equilateral triangle; but this case is not particularly interesting. Every cubic may be projected into a perfectly symmetrical form in this way:— the three real inflexions of the cubic lie on a certain straight line, and determine a point-cubic upon it; let the Hessian of this point-cubic be projected into the circular points at infinity. Then the cubic is symmetrically situated in respect of an equilateral asymptotic triangle.

The form in Fig. [58] may be called *inscribed*, that in Fig. [59] *escribed*. The inscribed cubic may have no oval nor double point, and then will lie entirely without the triangle.

Cubics with a crunode cannot be thus symmetrized; but their reciprocals can. In fact, every three-cusped quartic can be projected into a hypocycloid of three branches. For, any four points can be projected into any other four points; if then the three cusps and the intersection of the cuspidal tangents be projected respectively into the vertices and centre of an equilateral triangle, the thing is done. This proof is Prof. Cayley's. We learn in this way that, in any three-cusped quartic, the cuspidal tangents and the lines joining the cusps determine on the double tangent two point-cubics, whose common Hessian is the points of contact of that tangent.

Trinodal quartics have four double tangents, which are all real as lines when the nodes are real, but one is ideal (I borrow this convenient expression from Poncelet) or has imaginary contacts. If the three pairs of nodal tangents divide the ideal double tangent in the same anharmonic ratio, the quartic can be projected into a hypotrochoid, the contacts of the ideal tangent being then the circular points at infinity. The curve has many remarkable properties, which can be recognized at once from the symmetry of the projected figure.

Higher orders of symmetry are special cases. Thus quartic symmetry (as of a regular octogon) requires that the point-quartic shall be an harmonic system, so that its cubinvariant vanishes. Quintic Symmetry (regular pentagon) requires a point-quintic of the form ax^5+fy^5. The conditions that the quintic may be reduced to this form are that the invariants K of the eighth degree and L of the twelfth shall separately vanish (see Professor Sylvester's admirable *Trilogy*, *Phil. Trans.* 1864, p. 619).

The cube and sphere are examples of the Symmetry of the *cubangle*, whose projective property is that it determines on the plane at infinity a conjugate triad of the imaginary circle. One is naturally led to seek for the projective property of a regular tetrahedron. It determines on the plane at infinity four straight lines x, y, z, w; and if we assume the identical relation

$$x+y+z+w\equiv 0,$$

then the equation of the imaginary circle is

$$x^2+y^2+z^2+w^2=0,$$

as is easily shewn by the consideration that each face is an equilateral triangle. Many interesting properties of this conic will be found proved in the solution to Question 1690 *.

* [*Mathematics from the Educational Times*, Vol. III. pp. 92—96.]

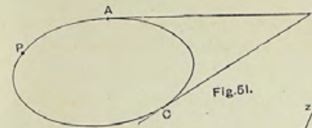


Fig. 51.

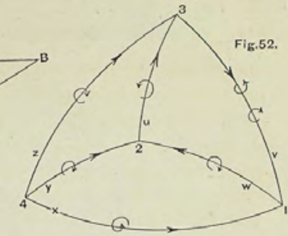


Fig. 52.

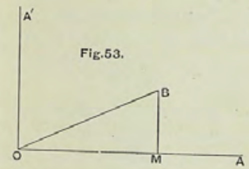


Fig. 53.

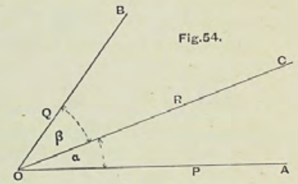


Fig. 54.

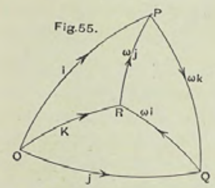


Fig. 55.

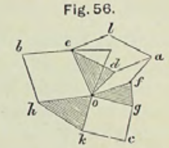


Fig. 56.

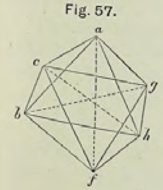


Fig. 57.

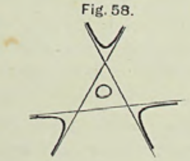


Fig. 58.

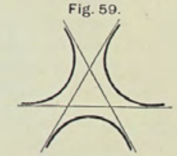


Fig. 59.

XLIX.

ON SOME EXTENSIONS OF THE FUNDAMENTAL PROPOSITION IN M. CHASLES'S THEORY OF CHARACTERISTICS*.

I MEAN by the "fundamental proposition" the following, viz.:—"If a variable system of two points on a right line be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions; then there are $a+b$ points on the right line at which the system of two points coalesces into one point."

This principle has been admirably extended by Dr Salmon to the case of two dimensions, thus:—"If a variable system of two points in a plane be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions, and that p pairs of points, each constituting a position of the system, may be found upon an arbitrary right line; then there are $a+b+p$ points in the plane at which the system of two points coalesces into one point."

The principle admits of further extension in two directions. First, we may consider a system of more than two points; and secondly, we may consider the systems as subject to a less number of relations than is sufficient to determine a single point. We are thus led to the following propositions:—

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point

* [From *Mathematics from the Educational Times*, Vol. v. pp. 49, 50.]

is determined to lie on a surface of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a surface of order b, and so on; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a surface of order $a+b+\dots$

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a curve of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a curve of order b, and so on, and that when all but the first two points are taken arbitrarily there are on an arbitrary right line p pairs of points each constituting a position of the first two points, and that $q, r\dots$ are the corresponding numbers for the other pairs of points of the system; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a curve of order $a+b+\dots+p+q+r+\dots$

It is not worth while to state the analogous propositions for Geometry of one and two dimensions, or the correlative propositions for lines and planes. I go on to exemplify the application of these propositions.

Let us begin with Mr Thomson's cubic (*Reprint*, Vol. II. p. 57). A conic is inscribed in a triangle so that the normals at the points of contact meet in a point; it is required to find the locus of this point. Consider now a variable system of three points, subject only to this condition; that if perpendiculars be drawn from them respectively to the three sides of a triangle, a conic may be drawn touching the sides of the triangle at the feet of those perpendiculars. Then, if we take two of the points arbitrarily, we determine two of the points of contact of the inscribed conic; that is we determine the conic itself uniquely, and therefore the third point of contact; and the normal at this point is therefore the locus of the third point of the system. That is to say, we have a variable system of three points so related that when any two of the points are taken arbitrarily, the locus of the third is a straight line;

consequently there are points in the plane at which the system of three points coalesces into one point (that is, where the three normals meet in a point), and the locus of such points is a curve of order $(1+1+1=)\ 3$.

To complicate the question, let us suppose that a conic is drawn to touch three given *conics*, so that the normals at the points of contact may meet in a point. Here, as before, we take our variable system of three points, one on each of the three normals. Take two of the points arbitrarily; from each of these we can draw four normals to the corresponding conic. Pairing these together, we have 16 pairs of points of contact. Now when we have given two tangents and their points of contact, the number of conics of the system which can be drawn to touch a given conic is 4. By determining two points of our variable system we have, therefore, determined 64 conics; on the third *given* conic, these determine 64 points of contact, and the normals through these may be held to constitute a curve of the 64th order. Thus we have a variable system of three points so related that when any two of them are taken arbitrarily, the third is determined to lie on a curve of the 64th order; consequently the locus of those points at which the system coalesces into one point, or the three normals meet in a point, is a curve of the order $(3\times 64)=192$. More generally, if we substitute for the variable conic a curve of order m, class n, and deficiency D, or say a curve of *species* (m, n, D), and for the three fixed conics $\frac{1}{2}(m+n-D+2)$ curves of orders $m_1, m_2, \dots$ and of classes $n_1, n_2, \dots$, then the corresponding locus will be of the order $3\phi(m, n, D).(m_1+n_1)(m_2+n_2)\dots$ where $\phi(m, n, D)$ is the number of curves of species (m, n, D) which can be drawn through $(m+n-D+1)$ points, or touching $(m+n-D+1)$ lines.

For another example, let us find the locus of those points the feet of the perpendiculars from which to four lines or planes in space are coplanar. In both these cases the locus comes out primarily of the fourth order; but the plane at infinity is evidently a part of the locus, the remainder of which is thus of the third order. In both cases the envelope of the plane through the feet of the perpendiculars is of the fourth class, and touches the plane at infinity. I conjecture that the imaginary circle is a curve of contact.

If a conicoid be drawn to touch five straight lines, so that the normal planes at the points of contact meet in a point, the locus of this point is of the tenth order. And so on *ad libitum*.

L*.

INSTRUMENTS USED IN MEASUREMENT.

BY *Measurement*, for scientific purposes, is meant the measurement of *quantities*. In each special subject there are quantities to be measured; and these are very various, as may be seen from the following list of those belonging to geometry and dynamics.

Geometrical Quantities.

Lengths
Areas
Volumes
Angles (plane and solid)
Curvatures (plane and solid)
Strains (elongation, torsion, shear).

Circumstances of Motion.	*Properties of Bodies.*
Time	Mass
Velocity	Weight
Momentum	Density
Acceleration	Specific gravity
Force	Elasticity (of form and volume)
Work	Viscosity
Horse-power	Diffusion
Temperature	Surface tension
Heat	Specific heat.

* [*Handbook to the Special Loan Collection of Scientific Apparatus*, 1876, L. pp. 55—59, LI. pp. 60—77. The object of L., LI. and other introductory Notices in the "Handbook" was to "make the Exhibition as useful and interesting as possible."]

Notwithstanding the very different characters of these quantities, they are all measured by reducing them to the same kind of quantity, and estimating that in the same way. Every quantity is measured by finding a *length* proportional to the quantity, and then measuring this length. This will, perhaps, be better understood if we consider one or two examples.

The measurement of *angles* occurs in a very large majority of scientific instruments. It is always effected by measuring the *length of an arc* upon a graduated circle; the circumference of this circle being divided not into inches or centimeters, but into degrees and parts of a degree—that is, into aliquot parts of the whole circumference.

As a step towards their final measurement, some quantities, of which work is a good instance, are represented in the form of *areas;* and there seems reason to believe that this method is likely to be extended. Instruments for measuring areas are called Planimeters; and one of the simplest of these is Amsler's, consisting of two rods jointed together, the end of one being fixed and that of the other being made to run round the area which is to be measured. The second rod rests on a wheel, which turns as the rod moves; and it is proved by geometry that the area is proportional to the distance through which the wheel turns. Thus the measurement of an area is reduced to the measurement of a length.

Volumes are measured in various ways, but all depending on the same principle. Quantities of earth excavated for engineering purposes are estimated by a rough determination of the shape of the cavity, and the measurement of its *dimensions,* namely, certain lengths belonging to it. The contents of a vessel are sometimes gauged in the same way; but the more accurate method is to fill it with liquid and then pour the liquid into a cylinder of known section, when the quantity is measured by the height of the liquid in the cylinder, that is, by a length. The volumes of irregular solids are also measured by immersing them in liquid contained in a uniform cylinder, and observing the height to which the liquid rises; that is, by measuring a length. An apparatus for this purpose is called a Stereometer. The liquid must be so chosen that no chemical action takes place between it and the solid immersed, and that it wets the solid so that no air bubbles adhere to the surface. Thus mercury is used in the case of metals by the Standards Department.

Time is measured for ordinary purposes by the length of the arc traced out by a moving hand on a circular clock-face. For astronomical purposes it is sometimes measured by counting the ticks of a clock which beats seconds, and estimating mentally the fractions of a second; and in cases where the period of an oscillation has to be found, it is determined by counting the number of oscillations in a time sufficient to make the number considerable, and then dividing that time by the number. But by far the most accurate way of measuring time is by means of the line traced by a pencil on a sheet of paper rolled round a revolving cylinder, or a spot of light moving on a sensitive surface. If the pencil is made to move along the length of the cylinder so as to indicate what is happening as time goes along, the time of each event will be found when the cylinder is unrolled by measuring the distance of the mark recording it from the end of the unrolled sheet, provided that the rate at which the cylinder goes round is known. In this way Helmholtz measured the rate of transmission of nerve-disturbance.

A very common case of the measurement of *force* is the barometer, which measures the pressure of the atmosphere per square inch of surface. This is determined by finding the height of the column of mercury which it will support (mercurial barometer), or the strain which it causes in a box from which the air has been taken out (aneroid barometer). The height in the former case may be measured directly, or it may first be converted into the quantity of turning of a needle, and then read off as length of arc on a graduated circle; in the latter case the strain is always indicated by a needle turning on a graduated circle.

The *mass,* and (what is proportional to it) the *weight,* of different bodies at the same place, are measured by means of a balance; and at first sight this mode of measurement seems different from those which we have hitherto considered. For we put the body to be weighed in one scale, and then put known weights into the other until equilibrium is obtained or the scale turns, and then we count the weights. But in a steelyard the weight is determined directly by means of a length; and in a balance which is accurate enough for scientific purposes, both methods are employed. We get as near as we can with the weights, and then the remainder is measured by a small rider of wire which is moved along the beam, and which determines the weight by its position; that is, by the measurement of a length.

For the measurement of weight in different places a spring-balance has to be used, and the weight is determined by the alteration it produces in the length of the spring; or else the length of the seconds pendulum is measured, from which the force of gravity on a given mass can be calculated. This last is an example of a very common and useful mode of measuring forces called into play by displacement or strain; namely, by measuring the period of the oscillations which they produce.

It seems unnecessary to consider any further examples, as all other quantities are measured by means of some simple geometrical or dynamical quantity which is proportional to them; as temperature by the height of mercury in a thermometer, heat by the quantity of ice it will melt (the volume of the resulting water), electric resistance by the length of a standard wire which has an equivalent resistance. It only remains to show how, when a length has been found proportional to the quantity to be measured, this length itself is measured.

For rough purposes, as for example in measuring the length of a room with a foot-rule, we apply the rule end on end, and count the number of times. For the piece left, we should apply the rule to it and count the number of inches. Or if we wanted a length expressed roughly for scientific purposes, we should describe it in metres or centimetres. But if it has to be expressed with greater accuracy, it must be described in hundredth, or thousandth, or millionth parts of a millimetre; and this is still done by comparing it with a scale.

But in order to estimate a length in terms of these very small quantities, it must be *magnified;* and this is done in three ways. First, geometrically, by what is called a vernier scale. This is a movable scale, which gains on the fixed one by one-tenth in each division. To measure any part of a division, we find how many divisions it takes the vernier to gain so much as that part; this is how many tenths the part is. The quantity to be measured is here geometrically multiplied by ten. Next, optically, by looking at the length and scale with a microscope or telescope. Third, mechanically, by a screw with a disc on its head, on which there is a graduated rim, called a micrometer screw. If the pitch of the screw is one-tenth and the radius of the disc ten times that of the screw, the motion is multiplied by one hundred. The two latter modes are combined together in an instrument called a micrometer-microscope. Another mechanical multiplier is a mirror which turns round and reflects light on a screen at some distance, as in Thomson's reflecting galvanometer.

Properly speaking, however, any description of a length by counting of standard lengths is imperfect and merely approximate. The true way of indicating a length is to draw a straight line which represents it on a fixed scale. And this is done by means of self-recording instruments, which measure lengths from time to time on a cylinder in the manner described above. It is only by this graphical representation of quantities that the laws of their variation become manifest, and that higher branch of measurement becomes possible which determines the nature of the connection between two simultaneously varying quantities.

LI.

INSTRUMENTS ILLUSTRATING KINEMATICS, STATICS, AND DYNAMICS.

Science of Motion.

GEOMETRY teaches us about the sizes, the shapes, and the distances of things; to know sizes and distances we have to measure *lengths*, and to know shapes we have to measure *angles*. The science of *Motion*, which is the subject of the present sketch, tells us about the changes in these sizes, shapes, and distances which take place from time to time. A body is said to move when it changes its place or position; that is to say, when it changes its distance from surrounding objects. And when the parts of a body move relatively to one another, *i.e.* when they alter their distance from one another, the body changes in size, or shape, or both. All these changes are considered in the science of motion.

Kinematics.

It is divided into two parts; the accurate description of motion, and the investigation of the circumstances under which particular motions take place. The description of motion may again be divided into two parts, namely, that which tells us *what* changes of position take place, and that which tells us *when* and *how fast* they take place. We might, for example, describe the motion of the hands of a clock, and say that they turn round on their axes at the centre of the clock-face in such a way that the minute-hand always moves twelve times as much as the hour-hand; this is the first part of the description of the motion. We might go on to say that when the clock is going correctly this motion takes place uniformly, so that the minute-

hand goes round once in each hour; and this would be the second part of the description. The first part is what was called Kinematics by Ampère; it tells us how the motions of the different parts of a machine depend on each other in consequence of the machinery which connects them. This is clearly an application of geometry alone, and requires no more measurements than those which belong to geometry, namely, measurements of lines and angles. But the name Kinematics is now conveniently made to include the second part also of the description of motion—when and how fast it takes place. This requires in addition the measurement of *time*, with which geometry has nothing to do. The word Kinematic is derived from the Greek *kinēma*, "motion;" and will therefore serve equally well to bear the restricted sense given it by Ampère, and the more comprehensive sense in which it is now used. And since the principles of this science are those which guide the construction not only of scientific apparatus, but of all instruments and machines, it may be advisable to describe in some detail the chief topics with which it deals.

Dynamics.

That part of the science which tells us about the circumstances under which particular motions take place is called *Dynamics*. It is found that the change of motion in a body depends on the position and state of surrounding bodies, according to certain simple laws; when considered as so depending on surrounding bodies, the rate of change in the quantity of motion is called *force*. Hence the name Dynamic, from the Greek *dynamis*, "force." The word *force* is here used in a technical sense, peculiar to the science of motion; the connection of this meaning with the meaning which the word has in ordinary discourse will be explained further on.

Statics and Kinetics.

Dynamics are again divided into two branches; the study of those circumstances in which it is possible for a body to remain at rest is called Statics, and the study of the circumstances of actual motion is called Kinetics. The simplest part of Statics, the doctrine of the Lever, was successfully studied before any other part of the science of motion, namely, by Archimedes, who proved that when a lever with unequal arms is balanced by weights at the ends of it, these weights are inversely proportional to the arms. But no real progress could be made in determining the conditions of rest, until the laws of actual motion had been studied.

Translation of Rigid Bodies.

Returning, then, to the description of motion, or Kinematics, we must first of all classify the different changes of position, of size, and of shape, with which we have to deal. We call a body *rigid* when it changes only its position, and not its size or shape, during the time in which we consider it. It is probable that every actual body is constantly undergoing slight changes of size and shape, even when we cannot perceive them; but in Kinematics, as in most other matters, there is a great convenience in talking about only one thing at a time. So we first of all investigate changes of position on the assumption that there are no changes of size and shape; or, in technical phrase, we treat of the motion of rigid bodies. Here an important distinction is made between motion in which the body merely travels from one place to another, and motion in which it also turns round. Thus the wheels of a locomotive engine not only travel along the line, but are constantly turning round; while the coupling-bar which joins two wheels on the same side remains always horizontal, though its changes of position are considerably complicated. A change of place in which there is no rotation is called a *translation*. In a rotation the different parts of the body are moving different ways, but in a translation all parts move in the same way. Consequently, in describing a translation we need only specify the motion of any one particle of the moving body; where by a *particle* is meant a piece of matter so small that there is no need to take account of the differences between its parts, which may therefore be treated for purposes of calculation as a point.

We are thus brought down to the very simple problem of describing the motion of a point. Of this there are certain cases which have received a great deal of attention on account of their frequent occurrence in nature; such as Parabolic Motion, Simple Harmonic Motion, Elliptic Motion. We propose to say a few words in explanation of each of these.

Parabolic Motion.

The motion of a *projectile*, that is to say, of a body thrown in any direction and falling under the influence of gravity, was investigated by Galileo; and this is the first problem of Kinetics that was ever solved. We must confine ourselves here to a description of the motion, without considering the way in which it depends on the circumstance of the presence of the earth at a certain distance from the moving body. Galileo found that the path of such a body, or the curve which it traces out, is a parabola; a curve which may be described as the shadow of a circle cast on a horizontal table by a candle which is just level with the highest point of the circle.

It is convenient to consider separately the vertical and the horizontal motion, for in accordance with a law subsequently stated in a general form by Newton, these two take place in complete independence of one another. So far as its horizontal motion is concerned, the projectile moves uniformly, as if it were sliding on perfectly smooth ice; and, so far as its vertical motion is concerned, it moves as if it were falling down straight. The nature of this vertical motion may be described in two ways, each of which implies the other. First, a falling body moves faster and faster as it goes down; and the rate at which it is going at any moment is strictly proportional to the number of seconds which has elapsed since it started. Thus its downward velocity is continually being added to at a uniform rate. Secondly, the whole distance fallen from the starting-point is proportional to the *square* of the number of seconds elapsed; thus, in three seconds a body will fall nine times as far as it will fall in one second. The latter of these statements was experimentally proved by Galileo; not, however, in the case of bodies falling vertically, which move too quickly for the time to be conveniently measured, but in the case of bodies falling down inclined planes, the law of which he at first assumed, and afterwards proved to be identical with that of the other. The former statement, that the velocity increases uniformly, is directly tested by an apparatus known as Attwood's machine, consisting essentially of a pulley, over which a string is hung with equal weights attached to its ends. A small bar of metal is laid on one of the weights, which begins to descend and pull the other one up; after a measured time the bar is lifted off, and then, both sides pulling equally, the motion goes on at the rate which had been acquired at that instant. The distance travelled in one second is then measured, and gives the velocity; this is found to be proportional to the time of falling with the bar on.

The second statement, that the space passed over is proportional to the square of the number of seconds elapsed, is verified by Morin's machine, which consists of a vertical cylinder which revolves uniformly while a body falling down at the side marks it with a pencil. The curve thus described is a record of the distance the body had fallen at every moment of time.

Fluxions.

This investigation of Galileo's was in more than one aspect the foundation of dynamical science; but not the least important of these aspects is the proof that either of the two ways of stating the law of falling bodies involves the other. Given that the distance fallen is proportional to the square of the time, to show that the velocity is proportional to the time itself; this is a particular case of the problem. Given where a body is at every instant, to find how fast it is going at every instant. The solution of this problem was given by Newton's Method of Fluxions. When a quantity changes from time to time, its *rate* of change is called the *fluxion* of the quantity. In the case of a moving body the quantity to be considered is the distance which the body has travelled; the fluxion of this distance is the rate at which the body is going. Newton's method solves the problem, Given *how big* a quantity is at

any time, to find its fluxion at any time. The method has been called on the Continent, and lately also in England, the Differential Calculus; because the difference between two values of the varying quantity is mentioned in one of the processes that may be used for calculating its fluxion. The inverse problem, Given that the velocity is proportional to the time elapsed, to find the distance fallen, is a particular case of the general problem, Given how fast a body is going at every instant, to find where it is at any instant; or, Given the fluxion of a quantity, to find the quantity itself. The answer to this is given by Newton's Inverse Method of Fluxions; which is also called the Integral Calculus, because in one of the processes which may be used for calculating the quantity, it is regarded as a whole (integer) made up of a number of small parts. The method of Fluxions, then, or Differential and Integral Calculus, takes its start from Galileo's study of parabolic motion.

Harmonic Motion.

The ancients, regarding the circle as the most perfect of figures, believed that circular motion was not only *simple*, that is, not made up by putting together other motions, but also *perfect*, in the sense that when once set up in perfect bodies it would maintain itself without external interference. The moderns, who know nothing about perfection except as something to be aimed at, but never reached, in practical work, have been forced to reject both of these doctrines. The second of them, indeed, belongs to Kinetics, and will again be mentioned under that head. But as a matter of Kinematics it has been found necessary to treat the uniform motion of a point round a circle as compounded of two oscillations. To take again the example of a clock, the extreme point of the minute-hand describes a circle uniformly; but if we consider separately its vertical position and its horizontal position, we shall see that it not only oscillates up and down, but at the same time swings from side to side, each in the same period of one hour. If we suppose a button to move up and down in a slit between the figures XII. and VI., in such a way as to be always at the same height at the end of the minute-hand, this button will have only one of the two oscillations which are combined in the motion of that point; and the other oscillation would be exhibited by a button constrained to move in a similar manner between the figures III. and IX., so as always to be either vertically above or vertically below the extreme point of the minute-hand. The laws of these two motions are identical, but they are so timed, that each is at its extreme position when the other is crossing the centre. An oscillation of this kind is called a *simple harmonic motion;* the name is due to Sir William Thomson, and was given on account of the intimate connection between the laws of such motions and the theory of vibrating strings. Indeed, the harmonic motion, simple or compound, is the most universal of all forms; it is exemplified not only in the motion of every particle of a vibrating solid, such as the string of a piano or violin, a tuning-fork, or the membrane of a drum, but in those minute excursions of particles of air which carry sound from one place to another, in the waves and tides of the sea, and in the amazingly rapid tremor of the luminiferous ether which, in its varying action on different bodies, makes itself known as light or radiant heat or chemical action. Simple harmonic motions differ from one another in three respects; in the extent or *amplitude* of the swing, which is measured by the distance from the middle point to either extreme; in the *period* or interval of time between two successive passages through an extreme position; and in the time of starting, or *epoch*, as it is called, which is named by saying what particular stage of the vibration was being executed at a certain instant of time. One of the most astonishing and fruitful theorems of mathematical science is this; that every *periodic* motion whatever, that is to say, every motion which exactly repeats itself again and again at definite intervals of time, is a compound of simple harmonic motions, whose periods are successively smaller and smaller aliquot parts of the original period, and whose amplitudes (after a certain number of them) are less and less as their periods are more rapid. The "harmonic" tones of a string, which are always heard along with the fundamental tone, are a particular case of

these constituents. The theorem was given by Fourier in connection with the flow of heat, but its applications are innumerable, and extend over the whole range of physical science.

The laws of combination of harmonic motions have been illustrated by some ingenious apparatus of Messrs Tisley and Spiller, and by a machine invented by Mr Donkin; but the most important practical application of these laws is to be found in Sir W. Thomson's Tidal Clock, and in a more elaborate machine which draws curves predicting the height of the tide at a given port for all times of the day and night with as much accuracy as can be obtained by direct observation. One special combination is worthy of notice. The union of a vertical vibration with a horizontal one of half the period gives rise to that figure of 8 which M. Marey has observed by his beautiful methods in the motion of the tip of a bird's or insect's wing.

Elliptic Motion.

The motion of the sun and moon relative to the earth was at first described by a combination of circular motions; and this was the immortal achievement of the Greek astronomers Hipparchus and Ptolemy. Indeed, in so far as these motions are periodic, it follows from Fourier's theorem mentioned above that this mode of description is mathematically sufficient to represent them; and astronomical tables are to this day calculated by a method which practically comes to the same thing. But this representation is not the simplest that can be found; it requires theoretically an infinite number of component motions, and gives no information about the way in which these are connected with one another. We owe to Kepler the accurate and complete description of planetary or elliptic motion. His investigations applied in the first instance to the orbit of the planet Mars about the sun, but it was found true of the orbits of all planets about the sun, and of the moon about the earth. The path of the moving body in each of these motions, is an ellipse, or oval shadow of a circle, a curve having various properties in relation to two internal points or foci, which replace as it were the one centre of a circle. In the case of the ellipse described by a planet, the sun is in one of these foci; in the case of the moon, the earth is one focus. So much for the geometrical description of the motion. Kepler further observed that a line drawn from the sun to a planet, or from the earth to the moon, and supposed to move round with the moving body, would sweep out equal areas in equal times. These two laws, called Kepler's first and second laws, complete the kinematic description of elliptic motion; but to obtain formulæ fit for computation, it was necessary to calculate from these laws the various harmonic components of the motion to and from the sun, and round it; this calculation has much occupied the attention of mathematicians.

The laws of rotatory motion of rigid bodies are somewhat difficult to describe without mathematical symbols, but they are thoroughly known. Examples of them are given by the apparatus called a gyroscope, and the motion of the earth; and an application of the former to prove the nature of the latter, made by Foucault, is one of the most beautiful experiments belonging entirely to dynamics.

Rotation.

Next in simplicity after the *translation* of a rigid body, come two kinds of motion which are at first sight very different, but between which a closer observation discovers very striking analogies. These are the motion of rotation about a fixed point, and the motion of sliding on a fixed plane. The first of these is most easily produced in practice by what is well known as a ball-and-socket joint; that is to say, a body ending in a portion of a spherical surface which can move about in a spherical cavity of the same size. The centre of the spherical surface is then a fixed point, and the motion is reduced to the sliding of one sphere inside another. In the same way, if we consider, for instance, the motion of a flat-iron on an ironing-board, we may see that this is not a pure translation, for the iron is frequently turned round as well as carried about; but the motion may be described as the sliding of one plane upon another. Thus in each case the matter to be studied is the sliding of one surface on another which it exactly fits. For

two surfaces to fit one another exactly, in all positions, they must be either both spheres of the same size, or both planes; and the latter case is really included under the former, for a plane may be regarded as a sphere whose radius has increased without limit. Thus, if a piece of ice be made to slide about on the frozen surface of a perfectly smooth pond, it is really rotating about a fixed point at the centre of the earth; for the frozen surface may be regarded as part of an enormous sphere, having that point for centre. And yet the motion cannot be practically distinguished from that of sliding on a plane.

In this latter case it is found that, excepting in the case of a pure translation, there is at every instant a certain point which is at rest, and about which as a centre the body is turning. This point is called the instantaneous centre of rotation; it travels about as the motion goes on, but at any instant its position is perfectly definite. From this fact follows a very important consequence; namely, that every possible motion of a plane sliding on a plane may be produced by the *rolling* of a curve in one plane upon a curve in the other. The point of contact of the two curves at any instant is the instantaneous centre at that instant. The problems to be considered in this subject are thus of two kinds: Given the curves of rolling to find the path described by any point of the moving plane; and, Given the paths described by *two* points of the moving plane (enough to determine the motion) to find the curves of rolling and the paths of all other points. An important case of the first problem is that in which one circle rolls on another, either inside or outside; the curves described by points in the moving plane are used for the teeth of wheels. To the second problem belongs the valuable and now rapidly increasing theory of *linkwork*, which, starting from the wonderful discovery of an exact parallel motion by M. Peaucellier, has received an immense and most unexpected development at the hands of Professor Sylvester, Mr Hart, and Mr A. B. Kempe.

Passing now to the spherical form of this motion, we find that the instantaneous centre of rotation (which is clearly equivalent to an instantaneous axis perpendicular to the plane) is replaced by an instantaneous axis passing through the common centre of the moving spheres. In the same way the rolling of one curve on another in the plane is replaced by the rolling of one *cone* upon another, the two cones having a common vertex at the same centre.

Analogous theorems have been proved for the most general motion of a rigid body. It was shown by M. Chasles that this is always similar to the motion of a corkscrew descending into a cork; that is to say, there is always a rotation about a certain instantaneous axis, combined with translation along this axis. The amount of translation per unit of rotation is called the *pitch* of the screw. The instantaneous screw moves about as the motion goes on, but at any given instant it is perfectly definite in position and pitch. And any motion whatever of a rigid body may be produced by the rolling and sliding of one surface on another, both surfaces being produced by the motion of straight lines. This crowning theorem in the geometry of motion is due to Professor Cayley. The laws of combination of screw motions have been investigated by Dr Ball.

Thus, proceeding gradually from the more simple to the more complex, we have been able to describe every change in the position of a body. It remains only to describe changes of size and shape. Of these there are three kinds, but they are all included under the same name—*strains*. We may have, first, a change of size without any change of shape, a uniform dilatation or contraction of the whole body in all directions, such as happens to a sphere of metal when it is heated or cooled. Next, we may have an elongation or contraction in one direction only, all lines of this body pointing in this direction being increased or diminished in the same ratio; such as would happen to a rod six feet long and an inch square, if it were stretched to seven feet long, still remaining an inch square. Thirdly, we may have a change of shape produced by the sliding of layers over one another, a mode of deformation which is easily produced in a pack of cards; this is called a *shear*. By appropriate combinations of these three, every change of size and shape may be produced; or we may even leave out the second element, and produce any strain whatever by a dilatation or contraction, and two shears.

Dynamics.

We have already said that the change of motion of a body depends upon the position and state of surrounding bodies. To make this intelligible it will be necessary to notice a certain property of the three kinds of motion of a point which we described.

The combination of velocities may be understood from the case of a body carried in any sort of cart or vehicle in which it moves about. The whole velocity of the body is then compounded of the velocity of the vehicle and of its velocity relative to the vehicle. Thus, if a man walks across a railway carriage his whole velocity is compounded of the velocity of the railway carriage and of the velocity with which he walks across.

When the velocity of a body is changed by adding to it a velocity in the same direction or in the opposite direction, it is only altered in amount; but when a transverse velocity is compounded with it, a change of direction is produced. Thus, if a man walks fore and aft on a steamboat, he only travels a little faster or slower; but if he walks across from one side to the other, he slightly changes the direction in which he is moving.

Now, in the parabolic motion of a projectile, we found that while the horizontal velocity continues unchanged, the vertical velocity increases at a uniform rate. Such a body is having a downwards velocity continually poured into it, as it were. This gradual change of the velocity is called *acceleration;* we may say that the acceleration of a projectile is always the same, and is directed vertically downwards.

In a simple harmonic motion it is found that the acceleration is directed towards the centre, and is always proportional to the distance from it. In the case of elliptic motion it was proved by Newton that the acceleration is directed towards the focus, and is inversely proportional to the square of the distance from it.

Let us now consider the circumstances under which these motions take place. To produce a simple harmonic motion we may take a piece of elastic string, whose length is equal to the height of a smooth table; then fasten one end of the string to a bullet and the other end to the floor, having passed it through a hole in the table, so that the bullet just rests on the top of the hole when the string is unstretched. If the bullet be now pulled away from the hole so that the string is stretched, and then let go, it will oscillate to and fro on either side of the hole with a simple harmonic motion. The acceleration (or rate of change of velocity) is here proportional to the distance from the hole; that is, to the *amount of elongation of the string*. It is directed towards the hole; that is, in the direction of this elongation. In the case of the moon moving round the earth, the acceleration is directed towards the earth, and is inversely proportional to the square of the distance from the earth.

In both these cases, then, the change of velocity depends upon surrounding circumstances; but in the case of the bullet, this circumstance is the strained condition of an adjoining body, namely, the elastic string; while in the case of the moon the circumstance is the position of a distant body, namely, the earth. The motion of a projectile turns out to be only a special case of the motion of the moon; for the parabola which it describes may be regarded as one end of a very long ellipse, whose other end goes round the earth's centre.

There is a remarkable difference between the two cases. The swing of the bullet depends upon its size; a large bullet will oscillate more slowly than a small one. This leads us to modify the rule. If a large bullet is equivalent to two small ones, then when it is going at the same rate it must contain twice as much motion as one of the small ones; or, as we now say, with the same velocity it has twice the *momentum*. Now the change of momentum is found to be the same for all bullets, when the momentum is reckoned as proportional to the quantity of matter in the bullet as well as to the velocity. The quantity of matter in a body is called its *mass;* for bodies of the same substance it is, of course, simply the quantity of that substance; but for bodies of different substances it is so reckoned as to make the rule hold good. The rule for this case may then be stated thus; the change of momentum of a body (that is, the change of velocity multiplied by the mass), depends

on the state of strain of adjoining bodies. Regarded as so depending, this change of momentum is called the *pressure* or *tension* of the adjoining body, according to the nature of the strain; both of these are included in the name *stress*, introduced by Rankine.

But in the case of projectiles, the acceleration is found to be the same for all bodies at the same place; and this rule holds good in all cases of planetary motion. So that it seems as if the change of velocity, and not the change of momentum, depended upon the position of distant bodies. But this case is brought under the same rule as the other by supposing that the mass of the moving body is to be reckoned among the "circumstances." The change of momentum is in this case called the attraction of gravitation, and we say that the attraction is proportional to the mass of the attracted body. And this way of representing the facts is borne out by the electrical and magnetic attractions and repulsions, where the change of momentum depends on the position and state of the attracting thing, and upon the electric charge or the induced magnetism of the attracted thing.

Force, then, is of two kinds; the stress of a strained adjoining body, and the attraction or repulsion of a distant body. Attempts have been made with more or less success to explain each of these by means of the other. In common discourse the word "force" means muscular effort exerted by the human frame. In this case the part of the human body which is in contact with the object to be moved is in a state of strain, and the force, dynamically considered, is of the first kind. But this state of strain is preceded and followed by nervous discharges, which are accompanied by the sensations of effort and of muscular strain; a complication of circumstances which does not occur in the action of inanimate bodies. What is common to the two cases is, that the change of momentum depends on the strain.

Having thus explained the law of Force, which is the foundation of Dynamics, we may consider the remaining laws of motion. It is convenient to state them first for particles, or bodies so small that we need take account only of their position.

Every particle, then, has a rate of change of momentum due to the position or state of every other particle, whether adjoining it or distant from it. These are compounded together by the law of composition of velocities, and the result of the whole is the actual change of momentum of the particle. This statement, and the law of Force stated above, amount together to Newton's first and second laws of motion. His third law is, that the change of momentum in one particle, due to the position or state of another, is equal and opposite to the change of momentum in the other, due to the position or state of the first.

By the help of these laws D'Alembert showed how the motion of rigid bodies, or systems of particles, might be dealt with. It appears from his method that two stresses, acting on a rigid body, may be equivalent, in their effect on the body as a whole, to a single stress, whose direction and position will be totally independent of the shape and nature of the body considered. The law of combination of stresses acting on a system of particles is, in fact, the same as the law of combination of velocities, so far as regards the motion of the system as a whole. This beautiful but somewhat complex result of Dynamics has been used in some text-books as the independent foundation of Statics, under the name of the parallelogram of forces; a singular inversion of the historical order and of the methods of the great writers.

When the result of all the circumstances surrounding a body is that there is no change of momentum, the body is said to be in equilibrium. In this case, if the body is at rest, it will remain so; and on this account the study of such conditions is called Statics. In dealing with the statics of rigid bodies, we have only to examine those cases in which the resultant of the external stresses and attractions acting on the body amounts to nothing. But the most important part of statics is that which finds the stresses acting in the interior of bodies between contiguous parts of them; for upon this depends the determination of the requisite strength of structures which have to bear given loads. It is found that the way in which the stress due to a given strain depends on the strain, varies according to the physical nature of the body; for bodies, however, which are not crystalline or fibrous, but which have the same properties in all directions, there are two quantities which, if known, will enable us always to calculate the stress due to a given strain. These are, the elasticity of volume, or resistance to change of size; and the rigidity, elasticity of figure, or resistance to change of shape. Problems relating to the interior state of bodies are far more difficult than those which regard them as rigid. Thus, if a beam is supported at its two ends, it is very easy to find the portion of its weight which is borne by each support; but the determination of the state of stress in the interior is a problem of great complexity.

There is one theorem of kinetics which must be mentioned here. If we multiply half the momentum of every particle of a body by its velocity, and add all the results together, we shall get what is called the kinetic energy of the body. When the body is moved from one position to another, if we multiply each force acting on it—whether attraction or stress—by the distance moved in the direction opposite to the force, and add the results, we shall get what is called the work done against the forces during the change of position. It does not at all depend on the rate at which the change is made, but only on the two positions. If a body moves, and loses kinetic energy, it does an amount of work equal to the kinetic energy lost. If it gains kinetic energy, an amount of work equal to this gain must be done to take it back from the new position to the old one. The amount of work which must be done to take a body from a certain standard position to the position which it has at present is called the potential energy of the body. The theorem may be stated in this form; the sum of the potential and kinetic energies is always the same, provided the surrounding circumstances do not alter. Hence the theorem is called the Conservation of Energy. It is one fact out of many that may be deduced from the equations of motion; it is not sufficient to determine the motion of a body, but it is exceedingly useful as giving a general result in cases where it might be difficult or undesirable to investigate all the particulars; and it is especially applicable to machines, the important question in regard to which is the amount of work which they can do.

It will have been seen that the science of motion depends on a few fundamental principles which are easily verified, and consists almost entirely of mathematical deductions and calculations based on those principles. It is no longer therefore an experimental science in the same sense as those are in which the fundamental facts are still being discovered. The apparatus connected with it may be conveniently classified under three heads:

(*a*) Apparatus for illustrating theorems or solving problems of kinematics, such as those mentioned above for compounding harmonic motions. There is reason to hope for great extension of our powers in this direction.

(*b*) Apparatus for measuring the dynamical quantities, such as weight, work, and the elasticities of different substances. These are more fully classified under Measurements.

(*c*) Apparatus designed for purposes belonging to other sciences, but illustrating by its structure and functions the results of kinematics or dynamics. In this class the remainder of the collection is included.

APPENDIX.