

Discrete approximations for determinantal point processes on continuum spaces and their applications

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Abstract

The determinantal point process, also called the fermion process, is formulated by Macchi(1975), Shirai-Takahashi(2000), and Soshnikov(2000). It appears in various mathematical systems such as uniform spanning trees, Schur measures, uniform lozenge tilings, the zeros of a hyperbolic Gaussian analytic function, the eigenvalue distribution of random matrices. It describes a natural structure of repulsive particle systems and has been extensively studied in the last two decades.

Determinant point processes are defined both on discrete sets and on continuum sets. Of the examples given above, the first three are point processes on discrete sets, and the latter are point processes on continuum sets. In this paper, we prove tail triviality, the Bernoulli property, and the Gibbs property for determinant point processes on continuum sets.

This thesis consists of four chapters. In Chapter 1, we give a brief introduction of this thesis. In Chapter 2, we introduce tree representation of α -determinantal point processes. The α -determinantal point process is a 1-parameter extension of the determinantal point process. Between μ and its tree representations, equations of correlation functions hold (Theorem 3.3.1). roughly speaking, that tree representations preserve regional information. We prove tail triviality in the case S is continuum from the result of the discrete case by combining tree representations and martingale convergence. In Chapter 3, we prove isomorphism between determinantal point processes with translation-invariant kernels and homogeneous Poisson point processes in the sense of measure-preserving dynamical systems due to Ornstein's theory and tree representations. In Chapter 4, we prove that for a point process μ on $\mathbb{R}^d$, the existence of logarithmic derivatives implies their Gibbs property in a weak sense. In the case $d = 1$, this implies μ has continuous local density. Due to Bufetov-Dymov-H.Osada(2019), the logarithmic derivative has calculated for a broad class of determinantal point process on $\mathbb{R}$. For example, the sine, Airy, Bessel, and other kernels related to de Branges spaces belong to the class. From this and our theorem, we obtain the closable of the Dirichlet form associated with the point processes.

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Chapter 1

Introduction

1.1 Determinantal point processes

Let S be a locally compact, second countable, and Hausdorff space. Then S is a Polish space, i.e., S is separable and have complete metrization. Denote $\text{Conf}(S)$ by the set of all nonnegative integer-valued Radon measures on S

$$\text{Conf}(S) = \left\{ \xi ; \xi = \sum_i \delta_{x_i}, \xi(K) < \infty \text{ for each compact } K \subset S \right\}.$$

We say a sequence of measures $\xi_n \in \text{Conf}(S)$ converges to ξ vaguely if for each $f \in C_0(S)$,

$$\lim_{n \rightarrow \infty} \int_S f d\xi_n = \int_S f d\xi.$$

The topology thus obtained on $\text{Conf}(S)$ is called the vague topology. Denote by $\mathfrak{B}_{\text{Conf}(S)}$ be the Borel σ -field. A configuration space over S is a measurable space $(\text{Conf}(S), \mathfrak{B}_{\text{Conf}(S)})$. A point process on S is a probability measure on a configuration space.

Let μ be a point process on S . Fix a Radon measure λ on S . A symmetric function ρ^m on S^m is called a m -point correlation function of a point process μ with respect to a reference measure λ if it satisfies

$$\int_{A_1^{k_1} \times \dots \times A_j^{k_j}} \rho^m(x_1, \dots, x_m) \lambda^{\otimes m}(dx) = \mathbb{E}^\mu \left[\prod_{i=1}^j \frac{\xi(A_i)}{(\xi(A_i) - k_i)!} \right].$$

Here, $A_1, \dots, A_j \subset S$ are disjoint and $k_1, \dots, k_j \in \mathbb{N}$ such that $k_1 + \dots + k_j = m$. If $\xi(A_i) \leq k_i$, then we set $\xi(A_i)/(\xi(A_i) - k_i)! = 0$.

A point process μ is called (K -)determinantal if its m -correlation functions are given by determinants of a kernel $K : S \times S \rightarrow \mathbb{C}$ such that

$$\rho^m(x_1, \dots, x_m) = \det[K(x_i, x_j)]_{i,j=1}^m. \quad (1.1.1)$$

The determinantal point process, also called the fermion process, is formulated by Macchi[12], Shirai and Takahashi[25] and Soshnikov[24]. Let $K : S \times S \rightarrow \mathbb{C}$ be a kernel function and denote by the same symbol the integral operator on $L^2(S, \lambda)$ such that for $f \in L^2(S, \lambda)$

$$Kf(x) = \int_S K(x, y)f(y)\lambda(dy).$$

Theorem 1.1.1 ([12, 25, 24]). *Let S be a locally compact, second countable, and Hausdorff space. Let λ be a Radon measure on S . Assume that $K : S \times S \rightarrow \mathbb{C}$ satisfies the follows.*

(A.1) *K is Hermitian symmetric.*

(A.2) *K is locally trace class.*

(A.3) *The spectrum of K is contained in $[0, 1]$.*

Then there exists a unique point process of which correlation functions are given by (1.1.1).

1.2 Tail triviality

Let S be a locally compact Hausdorff space with countable basis with metric $\mathbf{d}$. Fix a point $o \in S$ as the origin. Set $S_r = \{x \in S; \mathbf{d}(o, x) < r\}$. Assume that each S_r is relatively compact. Note that this notion depends on the choice of metric $\mathbf{d}$ on S .

For a Borel set A , we denote by $\pi_A : \text{Conf}(S) \rightarrow \text{Conf}(S)$ the projection of configuration such that $\xi(\cdot) \mapsto \xi(\cdot \cap A)$. Denote by $\text{Tail}_{\text{Conf}(S)}$ the tail σ -field such that

$$\text{Tail}_{\text{Conf}(S)} = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}].$$

Note that $\text{Tail}_{\text{Conf}(S)}$ is determined independently of the choice of $\mathbf{d}$. We say a point process μ on S is tail trivial if for all $A \in \text{Tail}_{\text{Conf}(S)}$,

$$\mu(A) \in \{0, 1\}.$$

In the case, S discrete, tail triviality for determinantal point processes was proved by Lyons[9]. Shirai and Takahashi[27] also proved under the restrictive assumption that the spectrum of K is contained in $(0, 1)$. In the viewpoint of ergodicity, tail triviality, also called the Kolmogorov property, implies strong mixing property of all orders.

In the case S is continuum, tail triviality is conjectured by Lyons[9] and proved by [20]. Tail triviality plays an important role in the proof of pathwise uniqueness of solutions of infinite-dimensional stochastic differential equations related to determinantal point processes [21].

1.3 Bernoullicity

An automorphism S of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a bi-measurable bijection such that $\mathbb{P} \circ S^{-1} = \mathbb{P}$. Let $S_G = \{S_g : g \in G\}$ be a group of automorphisms of $(\Omega, \mathcal{F}, \mathbb{P})$ parametrized by a group G . A measure-preserving dynamical system of G -action is the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$.

Example 1.3.1. *A typical example is a Bernoulli shift. Let $(S, \mathcal{S}, \mathbb{P})$ be a probability space. A $(G$ -action) Bernoulli shift is a quadruple of the direct product of probability space over a discrete group G and the canonical shift.*

Example 1.3.2. *In this paper, we consider translation invariant point processes on $\mathbb{Z}^d$ and $\mathbb{R}^d$. Homogeneous Poisson point processes on $\mathbb{R}^d$ with intensity $r > 0$ are typical ones.*

We say $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ is a factor of $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ if there exists a measurable map $\phi : \Omega \rightarrow \Omega'$ such that

$$\mathbb{P} \circ \phi^{-1} = \mathbb{P}', \quad \phi \circ S_g(x) = S'_g \circ \phi(x) \text{ for each } g \in G \text{ and a.s. } x \in \Omega.$$

We call ϕ the factor map. An isomorphism is a bi-measurable bijection ϕ between $\Omega_0 \subset \Omega$ and $\Omega'_0 \subset \Omega'$ such that $P(\Omega_0) = P'(\Omega'_0) = 1$ and both ϕ and ϕ^{-1} are factor maps. If there exists an isomorphism $\phi : \Omega \rightarrow \Omega'$, then $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ and $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ are said to be isomorphic. In this paper, we treat $\mathbb{R}^d$ - or $\mathbb{Z}^d$ -action systems.

We say $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{Z}^d})$ is Bernoulli if $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift. We say $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{R}^d})$ is Bernoulli if its restriction to $\mathbb{Z}^d$ -action $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{Z}^d})$ is Bernoulli.

For $\mathbb{Z}^d$ -action systems, the Bernoulli property implies tail triviality, strong mixing property of all orders, and ergodicity.

The Bernoulli property is important in the isomorphism problem. Ornstein proved any two Bernoulli shifts with the same entropy are isomorphic to each other. For a $\mathbb{Z}^d$ -action system $(\Omega, \mathcal{F}, \mathbb{P}, \mathbf{S}_{\mathbb{Z}^d})$, define its Kolmogorov-Sinai entropy by

$$h_{\mathbb{P}} = \sup_P \lim_{n \rightarrow \infty} \frac{1}{|R_n|} \sum_{A \in \bigvee_{g \in R_n} \mathbf{T}_g P} -\mathbb{P}(A) \log \mathbb{P}(A).$$

Here, $R_n = \{-n, \dots, 0\}^d \subset \mathbb{Z}^d$ and the sup is taken over all countable partition of $(\Omega, \mathcal{F})$ such that

$$\sum_{A \in P} -\mathbb{P}(A) \log(\mathbb{P}(A)) < \infty.$$

Remark that these logs in above definitions are binary logarithms. For $\mathbb{R}^d$ -action systems, define its entropy by the entropy of its restriction to $\mathbb{Z}^d$ -action.

It is known that the entropy is isomorphism invariance. Ornstein proved the converse holds for Bernoulli shifts.

Theorem 1.3.1 ([14, 15, 16, 17]). *Any two Bernoulli shifts are isomorphic if they have the same entropy.*

As a consequence of the general isomorphism theory, Poisson point processes are isomorphic to each other. This is a continuum version of the isomorphism theorem of Bernoulli shifts with infinite entropy.

Theorem 1.3.2 ([14, 15, 16, 17]). *Homogeneous Poisson point processes on $\mathbb{R}^d$ are isomorphic to each other regardless of their intensity.*

For determinantal point processes on $\mathbb{Z}^d$ with translation-invariant kernels, Lyons and Steif [11] and Shirai and Takahashi [27] independently proved the Bernoulli property. The latter gives a sufficient condition for the weak Bernoulli property in the case $\text{Spec}(K) \subset (0, 1)$. Remark that the weak Bernoulli property is stronger than the Bernoulli property. The former proved the Bernoulli property in the case $\text{Spec}(K) \subset [0, 1]$.

1.4 Gibbsianness

Let μ be a point process on $\mathbb{R}^d$. Let $\mu_{R, m, \eta}$ be a regular conditional probability given by

$$\mu_{R, m, \eta}(d\xi) = \mu(\pi_{B_R}(\cdot) \in d\xi \mid \xi(B_R) = m, \pi_{B_R^c} \xi = \pi_{B_R^c} \eta).$$

Here, $B_R \subset \mathbb{R}^d$ is the open ball of radius R centered at the origin and π_A is the projection of configuration on $A \subset \mathbb{R}^d$ such that $\xi(\cdot) \mapsto \xi(\cdot \cap A)$. Denote by Λ the Poisson point process with intensity 1.

A conventional definition of the canonical Gibbs measure is given by the Dobrushin-Lanford-Ruelle equation (1.4.1) (cf. [22, 23]). Let $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$. For $\xi = \sum_i \delta_{x_i}, \eta = \sum_j \delta_{y_j} \in \mathbf{Conf}(\mathbb{R}^d)$ and $R \in \mathbb{N}$, let

$$\mathcal{H}_{R,\eta}(\xi) = \sum_{x_i \in B_R} \Psi(x_i) + \sum_{i < j, x_i, x_j \in B_R} \Psi(x_i, x_j) + \sum_{x_i \in B_R, y_j \in B_R^c} \Psi(x_i, y_j).$$

Let $\Lambda_{R,m}$ be a conditional probability given by

$$\Lambda_{R,m}(d\xi) = \Lambda(\pi_R(\cdot) \in d\xi \mid \xi(B_R) = m).$$

We say μ is a canonical Gibbs measure for a free potential Φ and an interaction potential Ψ if μ satisfies the Dobrushin-Lanford-Ruelle equation

$$\mu_{R,m,\eta}(d\xi) = \frac{1}{Z} \exp(-\mathcal{H}_{R,\eta}(\xi)) \Lambda_{R,m}(d\xi) \quad (1.4.1)$$

for each $R, m \in \mathbb{N}$ and μ -a.s. η .

By replacing equality by inequality in (1.4.1), the quasi-Gibbs measure is introduced in [19]. For $\xi = \sum_i \delta_{x_i}$ and $R \in \mathbb{N}$, let

$$\mathcal{H}_R(\xi) = \sum_{x_i \in B_R} \Psi(x_i) + \sum_{i < j, x_i, x_j \in B_R} \Psi(x_i, x_j).$$

We say μ is a quasi-Gibbs measure for a free potential Φ and an interaction potential Ψ if μ satisfies the following inequality

$$Z_{R,m,\eta}^{-1} \exp(-\mathcal{H}_R(\xi)) \Lambda_{R,m}(d\xi) \leq \mu_{R,m,\eta}(d\xi) \leq Z_{R,m,\eta} \exp(-\mathcal{H}_R(\xi)) \Lambda_{R,m}(d\xi) \quad (1.4.2)$$

for each $R, m \in \mathbb{N}$ and μ -a.s. η .

Remark that the difference between (1.4.1) and (1.4.2) are not only equality and inequality but also parameter dependencies of normalizing constants and Hamiltonians.

We say μ is Gibbsian if, for each $R, m \in \mathbb{N}$ and μ -a.s. η , $\mu_{R,m,\eta}$ is absolutely continuous with respect to $\Lambda_{R,m}$.

Let $\mathbf{u}_m : (\mathbb{R}^d)^m \rightarrow \mathbf{Conf}(\mathbb{R}^d)$ be the delabeling map given by $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m \delta_{x_i}$. Define the symmetric measure $\check{\mu}_{R,m,\eta}$ on $(B_R)^m$ by the relation

$$\check{\mu}_{R,m,\eta} \circ \mathbf{u}_m^{-1} = \mu_{R,m,\eta}.$$

Then $\check{\mu}_{R,m,\eta}$ is a probability measure by construction.

Remark that μ is Gibbsian if, for each $\mathbb{R}, m \in \mathbb{N}$ and μ -a.s. η , $\check{\mu}_{R,m,\eta}$ is absolutely continuous with respect to the Lebesgue measure on $(B_R)^m$.

Our formulation of the Gibbs measure is weaker than the canonical Gibbs measure and the quasi Gibbs measure. However, due to remarks in Georgii-Yoo [6], existence of the Papangelou intensity is said to be Gibbsian in a general sense. On the other hand, for Gibbsian point processes, the continuity of Radon-Nikodym densities gives a sufficient condition for the closability of associated symmetric forms.

1.5 Dirichlet forms associated with point processes

For $R \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, set

$$\mathbf{Conf}(\mathbb{R}^d)_{R,m} = \{\xi \in \mathbf{Conf}(\mathbb{R}^d); \xi(B_R) = m\}.$$

Let $\mathbf{l}_{R,m} : \mathbf{Conf}(\mathbb{R}^d)_{R,m} \rightarrow (B_R)^m$ be a map such that

$$\mathbf{l}_{R,m}(\xi) = (\mathbf{l}_{R,m}^1(\xi), \mathbf{l}_{R,m}^2(\xi), \dots, \mathbf{l}_{R,m}^m(\xi))$$

and $\xi_R = \sum_{n=1}^m \delta_{\mathbf{l}_{R,m}^n(\xi)}$.

A function $\phi : \mathbf{Conf}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is called local if there exists a compact set $K \subset \mathbb{R}^d$ such that ϕ is $\sigma[\pi_K]$ -measurable. For a local function ϕ such that $\sigma[\pi_R]$ -measurable, we define symmetric functions $\phi_{R,m} : (B_R)^m \rightarrow \mathbb{R}$ by the relation

$$\phi_{R,m}(\mathbf{l}_{R,m}(\xi)) = \phi(\xi), \quad \xi \in \mathbf{Conf}(\mathbb{R}^d)_{R,m}. \quad (1.5.1)$$

Remark that $\phi_{R,m}$ is unique and $\phi(\xi) = \sum_{m=0}^{\infty} \phi_{R,m}(\mathbf{l}_{R,m}(\xi))$. Furthermore, $\phi_{R,m}$ is independent of the choice of R such that ϕ is $\sigma[\pi_R]$ -measurable.

A local function ϕ is said to be smooth if $\phi_{R,m}$ is smooth for each $R > Q$ and $m \in \mathbb{N}$. Here, Q is a positive number such that ϕ is $\sigma[\pi_Q]$ -measurable. Clearly, ϕ is smooth if $\phi_{R,m}$ is smooth for *some* $R > Q$ and each $m \in \mathbb{N}$.

Let $\mathcal{D}_\circ$ denote the space of all bounded local smooth functions on $\mathbf{Conf}(\mathbb{R}^d)$. Denote for $m \in \mathbb{N} \cup \{\infty\}$

$$\mathbf{D}_m[f, g](x) = \frac{1}{2} \sum_{n=1}^m \sum_{i=1}^d \partial_{i,n} f(x) \partial_{i,n} g(x).$$

Set $\mathbf{Conf}(\mathbb{R}^d)_m = \{\xi \in \mathbf{Conf}(\mathbb{R}^d); \xi(\mathbb{R}^d) = m\}$ for $m \in \mathbb{N} \cup \{\infty\}$. For $\phi, \psi \in \mathcal{D}_\circ$, we set $\mathbf{D}[\phi, \psi] : \mathbf{Conf}(\mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathbf{D}[\phi, \psi](\xi) &= \mathbf{D}_m[\phi_m(\mathbf{l}_m(\xi)), \psi_m(\mathbf{l}_m(\xi))] \quad \text{if } \xi \in \mathbf{Conf}(\mathbb{R}^d)_m, m \in \mathbb{N} \cup \{\infty\} \\ &= 0 \quad \text{if } \xi(\mathbb{R}^d) = 0. \end{aligned}$$

Here ϕ_m is defined in (1.5.1) and $\mathfrak{l}_m : \mathbf{Conf}(\mathbb{R}^d)_m \rightarrow (\mathbb{R}^d)^m$ is a map such that $\mathfrak{l}_m(\xi) = (\mathfrak{l}_m^1(\xi), \mathfrak{l}_m^2(\xi), \dots, \mathfrak{l}_m^m(\xi))$ and $\xi = \sum_{n=1}^m \delta_{\mathfrak{l}_m^n(\xi)}$. Set $(\mathcal{E}, \mathcal{D}) = (\mathcal{E}^\mu, \mathcal{D}^\mu)$ by

$$\begin{aligned}\mathcal{E}(\phi, \psi) &= \int_{\mathbf{Conf}(\mathbb{R}^d)} \mathbf{D}[\phi, \psi](\xi) \mu(d\xi), \\ \mathcal{D} &= \{\phi \in \mathcal{D}_\circ \cap L^2(\mathbf{Conf}(\mathbb{R}^d), \mu); \mathcal{E}(\phi, \phi) < \infty\}.\end{aligned}$$

Let $\mathfrak{u}_m : (\mathbb{R}^d)^m \rightarrow \mathbf{Conf}(\mathbb{R}^d)$ be the delabeling map given by $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m \delta_{x_i}$. Define the symmetric measure $\check{\mu}_{R,m,\eta}$ on $(B_R)^m$ by the relation

$$\check{\mu}_{R,m,\eta} \circ \mathfrak{u}_m^{-1} = \mu_{R,m,\eta}.$$

Theorem 1.5.1 ([18]). *Let μ be a point process on $\mathbb{R}^d$. Let*

$$\mathcal{E}_{R,m,\eta}(f, g) = \int_{(B_R)^m} \mathbf{D}_m[f, g](x) \check{\mu}_{R,m,\eta}(dx).$$

Assume that, for each $m, R \in \mathbb{N}$, $(\mathcal{E}_{R,m,\eta}, C_b^\infty((B_R)^m))$ is closable on $L^2((B_R)^m, \check{\mu}_{R,m,\eta})$ for μ -a.s. η . Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\mathbf{Conf}(\mathbb{R}^d), \mu)$.

Chapter 2

Tree representations of α -determinantal point processes

In this chapter, we introduce tree representations for α -determinantal point processes. The α -determinantal point processes is introduced in [26] as a one parameter extension of the determinantal point process. As its application, we prove tail triviality for determinantal point processes on continuum spaces.

2.1 α -determinantal point processes

Our aim is to introduce tree representations for α -determinantal point processes (also called the α -permanental point processes). Let S be a locally compact Hausdorff space with countable basis. We equip S with a Radon measure λ such that $\lambda(\mathcal{O}) > 0$ for any non-empty open set $\mathcal{O}$ in S . Let $\mathbf{S}$ be the configuration space over S (see (2.2.1) for definition). $\mathbf{S}$ is a Polish space equipped with the vague topology.

An α -determinantal point process μ on S is a probability measure on $(\mathbf{S}, \mathfrak{B}(\mathbf{S}))$ for which the m -point correlation function ρ^m with respect to λ is given by

$$\rho^m(\mathbf{x}) = \det_{\alpha}[\mathbf{K}(x_i, x_j)]_{i,j=1}^m. \quad (2.1.1)$$

Here $\mathbf{K}: S \times S \rightarrow \mathbb{C}$ is a measurable kernel, $\mathbf{x} = (x_1, \dots, x_m)$, and for $m \times m$ matrix $A = (a_{i,j})_{i,j=1}^m$

$$\det_{\alpha} A = \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \prod_{i=1}^m a_{i,\sigma(i)}, \quad (2.1.2)$$

where α is a real number, the summation is taken over the symmetric group $\mathfrak{S}_m$, the set of permutations of $\{1, 2, \dots, m\}$, and $\nu(\sigma)$ is the number of cycles of the permutation σ . μ is said to be α -determinantal point process associated with $(\mathbf{K}, \lambda)$.

The quantity (2.1.2) is called the α -determinant in [26] and also called the α -permanent in [29, 30]. For $\alpha = -1$, $\det_{-1} A$ is the usual determinant $\det A$ and μ is called a determinantal point process (also called a fermion point process). For $\alpha = 1$, $\det_1 A$ is the permanent $\text{per} A$ and μ is called a permanental point process (also called a boson point process). Letting α tend to 0, one obtain the Poisson point processes. Hence the α -determinantal point process is an one parameter extension of the determinantal point process.

We set $Kf(x) = \int_S K(x, y)f(y)\lambda(dy)$. We regard K as an operator on $L^2(S, \lambda)$ and denote it by the same symbol. We say K is of locally trace class if

$$K_A f(x) = \int 1_A(x)K(x, y)1_A(y)f(y)\lambda(dy)$$

is a trace class operator on $L^2(S, \lambda)$ for any compact set A . Throughout this paper, we assume:

(A1) $\alpha \in \{\frac{2}{m}; m \in \mathbb{N}\} \cup \{\frac{-1}{m}; m \in \mathbb{N}\}$. K is Hermitian symmetric and of locally trace class and $\text{Spec}(K) \subset [0, \infty)$. If $\alpha < 0$, $\text{Spec}(K) \subset [0, -\frac{1}{\alpha}]$.

From (A1) we deduce that the associated α -determinantal point process $\mu = \mu^{K, \lambda, \alpha}$ exists and is unique [26].

A λ -partition $\Delta = \{\mathcal{A}_i\}_{i \in I}$ of S is a countable collection of disjoint relatively compact, measurable subsets of S such that $\cup_i \mathcal{A}_i = S$ and that $\lambda(\mathcal{A}_i) > 0$ for all $i \in I$. For two partitions $\Delta = \{\mathcal{A}_i\}_{i \in I}$ and $\Gamma = \{\mathcal{B}_j\}_{j \in J}$, we write $\Delta \prec \Gamma$ if for each $j \in J$ there exists $i \in I$ such that $\mathcal{B}_j \subset \mathcal{A}_i$. We assume:

(A2) There exists a sequence of λ -partitions $\{\Delta(\ell)\}_{\ell \in \mathbb{N}}$ satisfying (2.1.3)–(2.1.5).

$$\Delta(\ell) \prec \Delta(\ell + 1) \quad \text{for all } \ell \in \mathbb{N}, \quad (2.1.3)$$

$$\sigma\left[\bigcup_{\ell \in \mathbb{N}} \mathfrak{F}_\ell\right] = \mathfrak{B}(S), \quad (2.1.4)$$

$$\#\{j; \mathcal{A}_{\ell+1, j} \subset \mathcal{A}_{\ell, i}\} = 2 \text{ for all } i \in I(\ell) \text{ and } \ell \in \mathbb{N}, \quad (2.1.5)$$

where we set $\Delta(\ell) = \{\mathcal{A}_{\ell, i}\}_{i \in I(\ell)}$ and $\mathfrak{F}_\ell := \mathfrak{F}_{\Delta(\ell)} = \sigma[\mathcal{A}_{\ell, i}; i \in I(\ell)]$.

Condition (2.1.5) is just for simplicity. This condition implies that the sequence $\{\Delta(\ell)\}_{\ell \in \mathbb{N}}$ has a binary tree-like structure. We remark that (A2) is a mild assumption and, indeed, satisfied if S is an open set in $\mathbb{R}^d$ and λ has positive density with respect to the Lebesgue measure.

Let $\mathfrak{G}_\ell$ be the sub- σ -field of $\mathfrak{B}(S)$ given by

$$\mathfrak{G}_\ell = \sigma[\{s \in S; s(\mathcal{A}_{\ell, i}) = n\}; i \in I(\ell), n \in \mathbb{N}]. \quad (2.1.6)$$

Combining (2.1.3) and (2.1.4) with (2.1.6), we obtain

$$\mathfrak{G}_\ell \subset \mathfrak{G}_{\ell+1}, \quad \sigma[\mathfrak{G}_\ell; \ell \in \mathbb{N}] = \mathfrak{B}(S).$$

Let $\mu(\cdot|\mathfrak{G}_\ell)$ be the regular conditional probability of μ with respect to $\mathfrak{G}_\ell$.

We can naturally regard $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$ as a discrete, countable set with the interpretation that each element $\mathcal{A}_{\ell,i}$ is a point. Thus, $\mu(\cdot|\mathfrak{G}_\ell)$ can be regarded as a point process on the discrete set $\Delta(\ell)$.

In Section 2.2 we introduce a sequence of *fiber bundle-like sets* $\mathbb{I}(\ell)$ ($\ell \in \mathbb{N}$) with base space $\Delta(\ell)$ with fiber consisting of a set of binary trees. We further expand $\mathbb{I}(\ell)$ to $\Omega(\ell)$ in (2.2.27), which has a fiber whose element is a product of a tree i and a component $\mathcal{B}_{\ell,i}$ of partitions. See notation after Theorem 2.2.1.

Let $\mu|_{\mathfrak{G}_\ell}$ denote the restriction of μ on $\mathfrak{G}_\ell$. By construction $\mu|_{\mathfrak{G}_\ell}(\mathbf{A}) = \mu(\mathbf{A}|\mathfrak{G}_\ell)$ for all $\mathbf{A} \in \mathfrak{G}_\ell$. In Theorem 2.2.1 and Theorem 2.2.2, we construct a lift $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ of $\mu|_{\mathfrak{G}_\ell}$ on the fiber bundle $\Omega(\ell)$ in (2.2.27).

The key point of the construction of the lift $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ is that we construct a consistent family of orthonormal bases $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ in (2.2.14) and (2.2.15), and introduce the kernel $\mathbf{K}_{\mathbb{F}(\ell)}$ on $\mathbb{I}(\ell)$ in (2.2.20) such that

$$\mathbf{K}_{\mathbb{F}(\ell)}(i, j) = \int_{S \times S} \mathbf{K}(x, y) f_{\ell,i}(x) f_{\ell,j}(y) \lambda(dx) \lambda(dy). \quad (2.2.20)$$

We shall prove in Lemma 2.3.2 that $\mathbf{K}_{\mathbb{F}(\ell)}$ is an α -determinantal kernel on $\mathbb{I}(\ell)$, and present $\nu_{\mathbb{F}(\ell)}$ as the associated α -determinantal point process on $\mathbb{I}(\ell)$. To some extent, $\nu_{\mathbb{F}(\ell)}$ is a Fourier transform of $\mu|_{\mathfrak{G}_\ell}$ through the orthonormal basis $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$. We shall prove in Theorem 2.2.1 that their correlation functions $\rho_{\mathfrak{G}_\ell}^m$ and $\rho_{\mathbb{F}(\ell)}^m$ satisfy a kind of Parseval's identity:

$$\int_{\mathbb{A}} \rho_{\mathfrak{G}_\ell}^m(\mathbf{x}) \lambda^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}), \quad (2.2.26)$$

which is a key to construct the lift $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$.

Vere-Jones [29, 30] introduced α -permanent (we call it α -determinant as refereed in [26]) as the coefficients which arise in expanding fractional powers of the characteristic polynomial of a matrix. Shirai-Takahashi [26] introduced the α -determinantal point processes. Their correlation functions are given by α -determinants of a kernel function. In the case $\alpha = -1$, the associated point process is the determinantal point processes [7, 9, 10, 24, 26, 27]. The condition (A1) is a part sufficient condition for the existence and uniqueness of α -determinantal point process in [26].

In [20], we introduced the tree representations for determinantal point processes on a continuum space under the assumption (A1) in the case $\alpha = -1$ and proved tail triviality by applying it. In this paper, we prove that the tree representations work for the α -determinantal point processes. Most statements in this paper are then the same as [20] except for the range of α . In particular, Lemma 2.3.1 and Lemma 2.3.3 correspond to Lemma 1 and Lemma 3 in [20], respectively.

The key idea is that $\mathbf{K}_{\mathbb{F}(\ell)}$ in (2.2.20) is given by a unitary operator $U : L^2(S, \lambda) \rightarrow L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$ such that $\mathbf{K} = U\mathbf{K}_{\mathbb{F}(\ell)}U^{-1}$. Hence $\mathbf{K}_{\mathbb{F}(\ell)}$ has the same spectrum of $\mathbf{K}$ and satisfies (A1).

The organization of the paper is as follows. In Section 2.2, we give definitions and concepts and state the main theorems (Theorems 2.2.1–2.2.3). We give tree representations of μ . In Section 2.3, we prove Theorem 2.2.1. In Section 2.4, we prove Theorem 2.2.2 and Theorem 2.2.3.

2.2 Tree representations

In this section, we recall various essentials and present the main theorems Theorem 2.2.1–Theorem 2.2.3.

A configuration space $\mathbf{S}$ over S is a set consisting of configurations on S such that

$$\mathbf{S} = \{\mathbf{s}; \mathbf{s} = \sum_i \delta_{s_i}, s_i \in S, \mathbf{s}(K) < \infty \text{ for any compact } K\}. \quad (2.2.1)$$

A probability measure μ on $(\mathbf{S}, \mathfrak{B}(\mathbf{S}))$ is called a point process, also called random point field. A symmetric function ρ^m on S^m is called the m -point correlation function of a point process μ with respect to a Radon measure λ if it satisfies

$$\int_{\mathbf{S}} \prod_{i=1}^j \frac{\mathbf{s}(A_i)!}{(\mathbf{s}(A_i) - k_i)!} \mu(d\mathbf{s}) = \int_{A_1^{k_1} \times \dots \times A_j^{k_j}} \rho^m(\mathbf{x}) \lambda^m(d\mathbf{x}). \quad (2.2.2)$$

Here $A_1, \dots, A_j \in \mathfrak{B}(S)$ are disjoint and $k_1, \dots, k_j \in \mathbb{N}$ such that $k_1 + \dots + k_j = m$. If $\mathbf{s}(A_i) - k_i \leq 0$, we set $\mathbf{s}(A_i)!/(\mathbf{s}(A_i) - k_i)! = 0$.

Let $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$ be as in (A2), where $\ell \in \mathbb{N}$. We set $\Delta = \{\mathcal{A}_i\}_{i \in I}$ such that

$$\Delta = \Delta(1), \quad \mathcal{A}_i = \mathcal{A}_{1,i} \quad I = I(1).$$

In consequence of (2.1.5), we assume without loss of generality that each element i of the parameter set $I(\ell)$ is of the form

$$I(\ell) = I \times \{0, 1\}^{\ell-1}. \quad (2.2.3)$$

That is, each $i \in I(\ell)$ is of the form $i = (j_1, \dots, j_\ell) \in I \times \{0, 1\}^{\ell-1}$. We take a label $i \in \cup_{\ell=1}^\infty I(\ell)$ in such a way that, for $\ell < \ell'$, $i \in I(\ell)$, and $i' \in I(\ell')$,

$$\mathcal{A}_{\ell,i} \supset \mathcal{A}_{\ell',i'} \Leftrightarrow i = (j_1, \dots, j_\ell) \text{ and } i' = (j_1, \dots, j_\ell, \dots, j_{\ell'}).$$

We denote by $\widetilde{\mathbb{I}}$ the set of all such parameters:

$$\widetilde{\mathbb{I}} = \sum_{\ell=1}^{\infty} I(\ell) = \sum_{\ell=1}^{\infty} I \times \{0, 1\}^{\ell-1}. \quad (2.2.4)$$

We can regard $\widetilde{\mathbb{I}}$ as a collection of binary trees and I is the set of their roots.

For $i = (j_1, \dots, j_\ell) \in \widetilde{\mathbb{I}}$, we set $\text{rank}(i) = \ell$. For i with $\text{rank}(i) = \ell$, we set

$$\mathcal{B}_i = \begin{cases} \mathcal{A}_{1,i} & \ell = 1, \\ \mathcal{A}_{\ell-1,i^-} & \ell \geq 2, \end{cases} \quad (2.2.5)$$

where $i^- = (j_1, \dots, j_{\ell-1})$ for $i = (j_1, \dots, j_\ell) \in I(\ell)$. Let $\mathbb{I} \subset \widetilde{\mathbb{I}}$ such that

$$\mathbb{I} = I + \sum_{\ell=2}^{\infty} \{i \in I(\ell); j_\ell = 0\}, \quad (2.2.6)$$

where $i = (j_1, \dots, j_\ell) \in I(\ell)$.

Let $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$ be an orthonormal basis of $L^2(S, \lambda)$ satisfying

$$\sigma[f_i; i \in \mathbb{I}, \text{rank}(i) = \ell] = \mathfrak{F}_\ell \quad \text{for each } \ell \in \mathbb{N}, \quad (2.2.7)$$

$$\text{supp}(f_i) = \mathcal{B}_i \quad \text{for each } i \in \mathbb{I}, \quad (2.2.8)$$

$$f_i(x) = 1_{\mathcal{A}_i}(x) / \sqrt{\lambda(\mathcal{A}_i)} \quad \text{for } \text{rank}(i) = 1. \quad (2.2.9)$$

For a given sequence of λ -partitions satisfying (A2), such an orthonormal basis exists. We present here an example.

Example 2.2.1 (Haar functions). *Typically we can take $S = \mathbb{R}$, $\lambda(dx) = dx$, and $I = \mathbb{Z}$. For $i = (j_1, \dots, j_\ell) \in I(\ell)$, we set $J_{1,i} = j_1$ and, for $\ell \geq 2$,*

$$J_{\ell,i} = j_1 + \sum_{n=1}^{\ell-1} \frac{j_n}{2^n}. \quad (2.2.10)$$

We take $\mathcal{A}_{\ell,i} = [J_{\ell,i}, J_{\ell,i} + 2^{-\ell+1})$.

Let $i = (j_1, \dots, j_\ell) \in \mathbb{I}$. We set for, $\ell = 1$ and $i = (j_1)$,

$$f_i(x) = 1_{[j_1, j_1+1)}(x)$$

and, for $\ell \geq 2$ and $i = (j_1, \dots, j_\ell) \in \mathbb{I}$,

$$f_i(x) = 2^{(\ell-1)/2} \{1_{[J_{\ell,i}, J_{\ell,i}+2^{-\ell+1})}(x) - 1_{[J_{\ell,i}+2^{-\ell+1}, J_{\ell,i}+2^{-\ell+2})}(x)\}.$$

We can easily see that $\{f_i\}_{i \in \mathbb{I}}$ is an orthonormal basis of $L^2(\mathbb{R}, dx)$. We remark that $j_\ell = 0$ because $i = (j_1, \dots, j_\ell) \in \mathbb{I}$ as we set in (2.2.6).

We next introduce the ℓ -shift of above objects such as $\mathbb{I}$, $\mathcal{B}_i$, and $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$. Let $\widetilde{\mathbb{I}}(1) = \mathbb{I}$ and, for $\ell \geq 2$, we set

$$\widetilde{\mathbb{I}}(\ell) := \sum_{r=1}^{\infty} I(\ell) \times \{0, 1\}^{r-1}, \quad (2.2.11)$$

where $I(\ell) = I \times \{0, 1\}^{\ell-1}$ is as in (2.2.3). For $\ell, r \in \mathbb{N}$, we set $\theta_{\ell-1, r} : \widetilde{\mathbb{I}} \rightarrow \widetilde{\mathbb{I}}(\ell)$ such that $\theta_{0, r} = \text{id}$ ($\ell = 1$) and, for $\ell \geq 2$,

$$\theta_{\ell-1, r}((j_1, \dots, j_{\ell+r-1})) = (\mathbf{j}_\ell, j_{\ell+1}, \dots, j_{\ell+r-1}) \in I(\ell) \times \{0, 1\}^{r-1}, \quad (2.2.12)$$

where $\mathbf{j}_\ell = (j_1, \dots, j_\ell) \in I(\ell)$. For $\ell = 1$, we set $\mathbb{I}(1) = \mathbb{I}$. For $\ell \geq 2$, we set

$$\mathbb{I}(\ell) = I(\ell) + \sum_{r=2}^{\infty} \theta_{\ell-1, r}(\mathbb{I}). \quad (2.2.13)$$

We set $\text{rank}(i) = r$ for $i \in I(\ell) \times \{0, 1\}^{r-1}$. By construction $\text{rank}(i) = r$ for $i \in \theta_{\ell-1, r}(\mathbb{I})$. Let $\mathbb{F}(\ell) = \{f_{\ell, i}\}_{i \in \mathbb{I}(\ell)}$ such that, for $r = \text{rank}(i)$,

$$f_{\ell, i}(x) = 1_{\mathcal{A}_{\ell, i}}(x) / \sqrt{\lambda(\mathcal{A}_{\ell, i})} \quad \text{for } r = 1, \quad (2.2.14)$$

$$f_{\ell, i}(x) = f_{\theta_{\ell-1, r}^{-1}(i)}(x) \quad \text{for } r \geq 2, \quad (2.2.15)$$

where $\Delta(\ell) = \{\mathcal{A}_{\ell, i}\}_{i \in I(\ell)}$ is given in (A2). Then $\mathbb{F}(\ell) = \{f_{\ell, i}\}_{i \in \mathbb{I}(\ell)}$ is an orthonormal basis of $L^2(S, \lambda)$. This follows from assumptions (2.2.14) and (2.2.15) and the fact that $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$ is an orthonormal basis.

Remark 2.2.1. (1) We note that $f_{\ell, i} \in \mathbb{F}(\ell)$ is a newly defined function if $\text{rank}(i) = 1$, whereas $f_{\ell, i} \in \mathbb{F}(\ell)$ is an element of $\mathbb{F}$ if $\text{rank}(i) \geq 2$. In particular, we see that

$$\{f_{\ell, i}\}_{i \in \mathbb{I}(\ell), \text{rank}(i) \geq 2} \subset \{f_i\}_{i \in \mathbb{I}, \text{rank}(i) \geq 2}. \quad (2.2.16)$$

(2) Let $j = (j_1, \dots, j_{\ell+r-1}) \in \mathbb{I}$ and $i = (\mathbf{j}_\ell, j_{\ell+1}, \dots, j_{\ell+r-1}) \in \mathbb{I}(\ell)$. Then

$$j = \theta_{\ell-1, r}^{-1}(i).$$

Furthermore, $f_{\ell, i} \in \mathbb{F}(\ell)$ and $f_j \in \mathbb{F}$ satisfy $f_{\ell, i} = f_j$ for $r = \text{rank}(i) \geq 2$.

(3) By construction, we see that

$$\sigma[f_{\ell, i}; i \in \mathbb{I}(\ell), \text{rank}(i) = r] = \mathfrak{F}_{\ell-1+r} \quad \text{for each } \ell, r \in \mathbb{N}, \quad (2.2.17)$$

$$\text{supp}(f_{\ell, i}) = \mathcal{B}_{\ell, i} \quad \text{for all } i \in \mathbb{I}(\ell), \quad (2.2.18)$$

where we set, for $j = \theta_{\ell-1, r}^{-1}(i)$ such that $\text{rank}(i) = r$,

$$\mathcal{B}_{\ell, i} = \mathcal{B}_j. \quad (2.2.19)$$

Using the orthonormal basis $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$, we set $\mathbf{K}_{\mathbb{F}(\ell)}$ on $\mathbb{I}(\ell)$ by

$$\mathbf{K}_{\mathbb{F}(\ell)}(i, j) = \int_{S \times S} \mathbf{K}(x, y) f_{\ell,i}(x) f_{\ell,j}(y) \lambda(dx) \lambda(dy). \quad (2.2.20)$$

Let $\lambda_{\mathbb{I}(\ell)}$ be the counting measure on $\mathbb{I}(\ell)$. We shall prove in Lemma 2.3.2 that $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ satisfies (A1). Hence we obtain the associated α -determinantal point process $\nu_{\mathbb{F}(\ell)}$ on $\mathbb{I}(\ell)$ from general theory [26].

For $i \in \mathbb{I}(\ell)$, let $\lambda_{f_{\ell,i}}(dx)$ be the probability measure on S such that

$$\lambda_{f_{\ell,i}}(dx) = |f_{\ell,i}(x)|^2 \lambda(dx). \quad (2.2.21)$$

For $\mathbf{i} = (i_n)_{n=1}^m \in \mathbb{I}(\ell)^m$ and $\mathbf{x} = (x_n)_{n=1}^m$, where $m \in \mathbb{N} \cup \{\infty\}$, we set

$$\lambda_{f_{\ell}, \mathbf{i}}(d\mathbf{x}) = \prod_{n=1}^m |f_{\ell,i_n}(x_n)|^2 \lambda(dx_n). \quad (2.2.22)$$

By (2.2.15) $\lambda_{f_{\ell}, \mathbf{i}}$ is a probability measure on S^m . By (2.2.18), we have

$$\lambda_{f_{\ell}, \mathbf{i}}\left(\prod_{n=1}^m \mathcal{B}_{\ell,i_n}\right) = 1. \quad (2.2.23)$$

Let $\mathfrak{F}_{\ell}^m = \sigma[\mathcal{A}_{\ell,i_1} \times \cdots \times \mathcal{A}_{\ell,i_m}; i_n \in \mathbb{I}(\ell), n = 1, \dots, m]$. Let $\mathfrak{G}_{\ell}$ be the sub- σ -field as in (2.1.6). An $\mathfrak{F}_{\ell}^m$ -measurable symmetric function $\rho_{\mathfrak{G}_{\ell}}^m$ on S^m is called the m -point correlation function of $\mu|_{\mathfrak{G}_{\ell}}$ with respect to λ if it satisfies

$$\int_S \prod_{i=1}^j \frac{s(A_i)!}{(s(A_i) - k_i)!} \mu(ds) = \int_{A_1^{k_1} \times \cdots \times A_j^{k_j}} \rho_{\mathfrak{G}_{\ell}}^m(\mathbf{x}) \lambda^m(d\mathbf{x}). \quad (2.2.24)$$

Here $A_1, \dots, A_j \in \mathfrak{F}_{\ell}$ are disjoint and $k_1, \dots, k_j \in \mathbb{N}$ such that $k_1 + \cdots + k_j = m$. If $s(A_i) - k_i \leq 0$, we set $s(A_i)!/(s(A_i) - k_i)! = 0$.

Let $\nu_{\mathbb{F}(\ell)}$ be the α -determinantal point process associated to $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ as before. Let $\rho_{\mathfrak{G}_{\ell}}^m$ and $\rho_{\mathbb{F}(\ell)}^m$ be the m -point correlation functions of $\mu|_{\mathfrak{G}_{\ell}}$ and $\nu_{\mathbb{F}(\ell)}$ with respect to λ and $\lambda_{\mathbb{I}(\ell)}$, respectively. We now state one of our main theorems:

Theorem 2.2.1. *Let $\mathbb{I}_{\ell}(\mathcal{A}) = \{i \in \mathbb{I}(\ell); \mathcal{B}_{\ell,i} \subset \mathcal{A}\}$. For $\mathbb{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$, we set*

$$\mathbb{I}_{\ell}(\mathbb{A}) = \mathbb{I}_{\ell}(\mathcal{A}_1) \times \cdots \times \mathbb{I}_{\ell}(\mathcal{A}_m). \quad (2.2.25)$$

Assume that $\mathcal{A}_n \in \Delta(\ell)$ for all $n = 1, \dots, m$. Then

$$\int_{\mathbb{A}} \rho_{\mathfrak{G}_{\ell}}^m(\mathbf{x}) \lambda^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_{\ell}(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}). \quad (2.2.26)$$

Let $\mathbb{I}(\ell)$ be the configuration space over $\mathbb{I}(\ell)$. Let

$$\Omega(\ell) := \bigcup_{i \in \mathbb{I}(\ell)} \{i\} \times \mathcal{B}_{\ell,i}. \quad (2.2.27)$$

Let $\underline{\Omega}(\ell)$ be the configuration space over $\Omega(\ell)$. Then by definition each element $\omega \in \underline{\Omega}(\ell)$ is of the form $\omega = \sum_n \delta_{(i_n, s_n)}$ such that $s_n \in \mathcal{B}_{\ell, i_n}$. Hence

$$\underline{\Omega}(\ell) = \{\omega = \sum_n \delta_{(i_n, s_n)}; i = \sum_n \delta_{i_n} \in \mathbb{I}(\ell), s_n \in \mathcal{B}_{\ell, i_n}\}. \quad (2.2.28)$$

We exclude the zero measure from $\underline{\Omega}(\ell)$.

Let $\lambda_{f_{\ell}, i}$ be as in (2.2.21). We set

$$\lambda_{\mathbb{F}(\ell)} = \prod_{i \in \mathbb{I}(\ell)} \lambda_{f_{\ell}, i}, \quad \lambda_{f_{\ell}, i} = \prod_n \lambda_{f_{\ell}, i_n}. \quad (2.2.29)$$

Remark 2.2.2. (1) A configuration $i \in \mathbb{I}(\ell)$ can be represented as $i = \sum_n \delta_{i_n}$ and this may have multiple points.

(2) Let $i \in \mathbb{I}(\ell)$. Suppose that for some $m \in \mathbb{N} \cup \{\infty\}$, i has plural representations such as

$$i = \sum_{n=1}^m \delta_{i_n} = \sum_{n=1}^m \delta_{j_n}.$$

Then $\prod_{n=1}^m \mathcal{B}_{\ell, i_n}$ and $\prod_{n=1}^m \mathcal{B}_{\ell, j_n}$ can be different subsets of S^m . However, the product probability spaces $(\prod_{n=1}^m \mathcal{B}_{\ell, i_n}, \lambda_{f_{\ell}, i})$ and $(\prod_{n=1}^m \mathcal{B}_{\ell, j_n}, \lambda_{f_{\ell}, i})$ are the same under the identification such that

$$\prod_{n=1}^m \mathcal{B}_{\ell, i_n} \ni (x_n)_{n=1}^m \mapsto (x_{\sigma(n)})_{n=1}^m \in \prod_{n=1}^m \mathcal{B}_{\ell, j_n}.$$

Here, σ is the permutation such that $i_{\sigma(n)} = j_n$. They do not depend on the representations of i under this identification.

We set $\iota_{\ell}: \underline{\Omega}(\ell) \rightarrow \mathbb{I}(\ell)$ such that $\sum_n \delta_{(i_n, s_n)} \mapsto \sum_n \delta_{i_n}$. For $i \in \mathbb{I}(\ell)$, let

$$\kappa_{\ell, i}: \{\omega \in \underline{\Omega}(\ell); \iota_{\ell}(\omega) = i\} \rightarrow \prod_n \mathcal{B}_{\ell, i_n}$$

such that $\sum_n \delta_{(i_n, s_n)} \mapsto (s_n)$. Let $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ be the probability measure on $\underline{\Omega}(\ell)$ given by the disintegration made of

$$(\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}) \circ \iota_{\ell}^{-1}(di) = \nu_{\mathbb{F}(\ell)}(di), \quad (2.2.30)$$

$$\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}(\kappa_{\ell, i}(\omega) \in d\mathbf{s} | \iota_{\ell}(\omega) = i) = \lambda_{f_{\ell}, i}(d\mathbf{s}), \quad \mathbf{s} = (s_n) \text{ for } i = \sum_n \delta_{i_n}. \quad (2.2.31)$$

Remark 2.2.3. (1) We can naturally regard the probability measures in (2.2.31) as a point process on $\prod_n \mathcal{B}_{\ell, i_n}$ supported on the set of configurations with exactly one particle configuration $\mathbf{s} = \delta_{\mathbf{s}}$ on $\prod_n \mathcal{B}_{\ell, i_n}$, that is, $\mathbf{s} = (s_n)$ is such that $s_n \in \mathcal{B}_{\ell, i_n}$.
 (2) We can regard $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ as a marked point process as follows: The configuration $\mathbf{i}$ is distributed according to $\nu_{\mathbb{F}(\ell)}$, while the marks are independent and for each $\mathbf{i}$ the mark $\mathbf{s}$ is distributed according to $\lambda_{f_{\ell, \mathbf{i}}}$. Thus the space of marks depends on $\mathbf{i}$.

Theorem 2.2.2. Let $\mathbf{u}_{\ell} : \underline{\Omega}(\ell) \rightarrow \mathbb{S}$ be such that $\mathbf{u}_{\ell}(\omega) = \sum_n \delta_{s_n}$, where $\omega = \sum_n \delta_{(i_n, s_n)}$. Then

$$\mu|_{\mathfrak{G}_{\ell}} = (\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}) \circ \mathbf{u}_{\ell}^{-1}|_{\mathfrak{G}_{\ell}}. \quad (2.2.32)$$

Remark 2.2.4. Theorem 2.2.2 implies that $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ is a lift of $\mu|_{\mathfrak{G}_{\ell}}$ onto $\underline{\Omega}(\ell)$. We can naturally regard $\tilde{\mathbb{I}}(\ell)$ as binary trees. Hence we call $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ a tree representation of μ of level ℓ .

We present a decomposition of $\mu|_{\mathfrak{G}_{\ell}}$, which follows from Theorem 2.2.2 immediately. Let $\lambda_{f_{\ell, \mathbf{i}}}^{\mathbf{u}} = \lambda_{f_{\ell, \mathbf{i}}} \circ \mathbf{u}_{\ell, \mathbf{i}}^{-1}$, where $\mathbf{u}_{\ell, \mathbf{i}} : \prod_n \mathcal{B}_{\ell, i_n} \rightarrow \mathbb{S}$ is the unlabel map such that

$$\mathbf{u}_{\ell, \mathbf{i}}((s_n)) = \sum_n \delta_{s_n}. \quad (2.2.33)$$

Theorem 2.2.3. For each $\mathbf{A} \in \mathfrak{G}_{\ell}$,

$$\mu(\mathbf{A}) = \int_{\mathbb{I}(\ell)} \nu_{\mathbb{F}(\ell)}(d\mathbf{i}) \lambda_{f_{\ell, \mathbf{i}}}^{\mathbf{u}}(\mathbf{A}). \quad (2.2.34)$$

We remark that $\mu|_{\mathfrak{G}_{\ell}}$ is *not* an α -determinantal point process. Hence we exploit $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ instead of $\mu|_{\mathfrak{G}_{\ell}}$. As we have seen in Theorem 2.2.2, $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ is a lift of $\mu|_{\mathfrak{G}_{\ell}}$ in the sense of (2.2.32), from which we can deduce nice properties of $\mu|_{\mathfrak{G}_{\ell}}$. Indeed, an application of Theorem 2.2.2 is tail triviality of μ in the case $\alpha = -1$ [20].

2.3 Proof of Theorem 2.2.1

The purpose of this section is to prove Theorem 2.2.1. In Lemma 2.3.1, we present a kind of Parseval's identity of kernels $\mathbf{K}$ and $\mathbf{K}_{\mathbb{F}(\ell)}$ using the orthonormal basis $\mathbb{F}(\ell)$, where $\mathbf{K}_{\mathbb{F}(\ell)}$ is the kernel given by (2.2.20) and $\mathbb{F}(\ell)$ is as in (2.2.14) and (2.2.15). In Lemma 2.3.2, we prove $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ is a determinantal kernel and the associated α -determinantal point process $\nu_{\mathbb{F}(\ell)}$ exists. We will lift the Parseval's identity between $\mathbf{K}$ and $\mathbf{K}_{\mathbb{F}(\ell)}$ to that of correlation functions of $\mu|_{\mathfrak{G}_{\ell}}$ and $\nu_{\mathbb{F}(\ell)}$ in Theorem 2.2.1.

By definition $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ satisfies

$$\int_S |f_{\ell,i}(x)|^2 \lambda(dx) = 1 \quad \text{for all } i \in \mathbb{I}(\ell), \quad (2.3.1)$$

$$\int_S f_{\ell,i}(x) f_{\ell,j}(x) \lambda(dx) = 0 \quad \text{for all } i \neq j \in \mathbb{I}(\ell). \quad (2.3.2)$$

Lemma 2.3.1. (1) Let $P(x) = \sum_i p(i) f_{\ell,i}(x)$ and $Q(y) = \sum_j q(j) f_{\ell,j}(y)$. Suppose that the supports of p and q are finite sets. Then

$$\int_{S \times S} \mathbf{K}(x, y) \overline{P(x)} Q(y) \lambda(dx) \lambda(dy) = \sum_{i,j} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \overline{p(i)} q(j). \quad (2.3.3)$$

(2) We have an expansion of $\mathbf{K}$ in $L^2_{\text{loc}}(S \times S, \lambda \times \lambda)$ such that

$$\mathbf{K}(x, y) = \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x) f_{\ell,j}(y). \quad (2.3.4)$$

(3) Let $\mathbb{I}(\ell; R) = \{i \in \mathbb{I}(\ell); \text{rank}(i) \leq R\}$, where $\text{rank}(i)$ is defined before (2.2.14). Let

$$\mathbf{K}_R(x, y) = \sum_{i,j \in \mathbb{I}(\ell; R)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x) f_{\ell,j}(y). \quad (2.3.5)$$

We set $\mathbb{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$. Assume that $\mathcal{A}_n \in \Delta(\ell)$ for $n = 1, \dots, m$. Then for $\sigma \in \mathfrak{S}_m$,

$$\lim_{R \rightarrow \infty} \int_{\mathbb{A}} \prod_{n=1}^m \mathbf{K}_R(x_n, x_{\sigma(n)}) \lambda^m(d\mathbf{x}) = \int_{\mathbb{A}} \prod_{n=1}^m \mathbf{K}(x_n, x_{\sigma(n)}) \lambda^m(d\mathbf{x}). \quad (2.3.6)$$

Proof. From (2.2.20) we deduce that

$$\begin{aligned} & \int_{S \times S} \mathbf{K}(x, y) \overline{P(x)} Q(y) \lambda(dx) \lambda(dy) \\ &= \int_{S \times S} \mathbf{K}(x, y) \overline{\sum_i p(i) f_{\ell,i}(x)} \sum_j q(j) f_{\ell,j}(y) \lambda(dx) \lambda(dy) \\ &= \sum_{i,j} \int_{S \times S} \mathbf{K}(x, y) f_{\ell,i}(x) f_{\ell,j}(y) \lambda(dx) \lambda(dy) \overline{p(i)} q(j) \\ &= \sum_{i,j} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \overline{p(i)} q(j). \end{aligned} \quad (2.3.7)$$

This yields (2.3.3). We have thus proved (1). By a direct calculation, we have

$$\begin{aligned} \int_S \overline{P(x)} f_{\ell,i}(x) \lambda(dx) &= \int_S \sum_j \overline{p(j)} f_{\ell,j}(x) f_{\ell,i}(x) \lambda(dx) = \overline{p(i)}, \\ \int_S Q(y) f_{\ell,j}(y) \lambda(dy) &= \int_S \sum_i q(i) f_{\ell,i}(y) f_{\ell,j}(y) \lambda(dy) = q(j). \end{aligned} \quad (2.3.8)$$

Combining (2.3.7) and (2.3.8) yields

$$\begin{aligned} \int_{S \times S} K(x, y) \overline{P(x)} Q(y) \lambda(dx) \lambda(dy) &= \\ \int_{S \times S} \sum_{i,j} K_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x) f_{\ell,j}(y) \overline{P(x)} Q(y) \lambda(dx) \lambda(dy). \end{aligned}$$

This implies (2.3.4).

Without loss of generality, we can assume σ is a cyclic permutation. We prove (2.3.6) only for $\sigma = (1, 2, \dots, m)$. Let $\mathcal{A}_n \in \Delta(\ell)$ for $n \geq 0$. Let $\mathbb{A} = \mathbb{A}(m) = \mathcal{A}_0 \times \dots \times \mathcal{A}_m$. For $0 \leq n \leq m$, we set

$$K^{\mathbb{A},n}(x, y) = \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_{n-1}} \prod_{p=1}^n K(x_{p-1}, x_p) \lambda(dx_1) \dots \lambda(dx_{n-1}), \quad (2.3.9)$$

$$K_R^{\mathbb{A},n}(x, y) = \int_{\mathcal{A}_{m-(n-1)} \times \dots \times \mathcal{A}_{m-1}} \prod_{p=1}^n K_R(x_{p-1}, x_p) \lambda(dx_1) \dots \lambda(dx_n). \quad (2.3.10)$$

where $x_0 = x$, $x_n = y$, $K^{\mathbb{A},0}(x, y) = K_R^{\mathbb{A},0}(x, y) = \delta_x(y)$, $K^{\mathbb{A},1}(x, y) = K(x, y)$, and $K_R^{\mathbb{A},1}(x, y) = K_R(x, y)$. By assumption K is a trace class operator on $L^2(\mathcal{B}, \lambda)$ for a relatively compact set $\mathcal{B}$ such that $\bigcup_{p=1}^m \mathcal{A}_p \subset \mathcal{B}$. Then $K^{\mathbb{A},n}$ is also a trace class operator on $L^2(\mathcal{B}, \lambda)$ for each $n \in \{1, \dots, m\}$. In particular, $K^{\mathbb{A},n}$ is a Hilbert-Schmidt operator on $L^2(\mathcal{B}, \lambda)$ and satisfies

$$\int_{\mathcal{B}^2} |K^{\mathbb{A},n}(x, y)|^2 \lambda(dx) \lambda(dy) < \infty. \quad (2.3.11)$$

We set for $k, n \geq 0$ such that $k + n = m$,

$$\mathbb{L}_R^{\mathbb{A},k,n}(x, y) = \int_{\mathcal{A}_k} K^{\mathbb{A},k}(x, z) K_R^{\mathbb{A},n}(z, y) \lambda(dz). \quad (2.3.12)$$

We shall prove the following by induction for m : for all $k, n \geq 0$ such that $k + n = m$ and for any $\mathbb{A} = \mathcal{A}_0 \times \dots \times \mathcal{A}_m$ such that $\mathcal{A}_p \in \Delta(\ell)$ for $p = 0, \dots, m$

$$\lim_{R \rightarrow \infty} \int_{\mathcal{A}_0 \times \mathcal{A}_m} |\mathbb{L}_R^{\mathbb{A},k,n}(x, y) - K^{\mathbb{A},m}(x, y)|^2 \lambda(dx) \lambda(dy) = 0, \quad (2.3.13)$$

$$\sup_R \int_{\mathcal{A}_0 \times \mathcal{A}_m} |\mathbb{L}_R^{\mathbb{A},k,n}(x, y)|^2 \lambda(dx) \lambda(dy) < \infty. \quad (2.3.14)$$

Let $m = 1$. For $(k, n) = (0, 1)$, Lemma 2.3.1 (2) implies (2.3.13) and (2.3.14). For $(k, n) = (1, 0)$, $\mathbb{L}_R^{\mathbb{A}, 1, 0}(x, y) = \mathbb{K}^{\mathbb{A}, 1}(x, y)$ by the definition in (2.3.12). Then (2.3.13) and (2.3.14) hold for $(k, n) = (1, 0)$. Hence (2.3.13) and (2.3.14) holds for $m = 1$.

Suppose (2.3.13) and (2.3.14) hold for $1, \dots, m-1$. Let $k+n = m-1$ and $\mathbb{A} = \mathcal{A}_0 \times \dots \times \mathcal{A}_m$. By a straightforward calculation,

$$\begin{aligned} & \mathbb{L}_R^{\mathbb{A}, k, n+1}(x, y) - \mathbb{L}_R^{\mathbb{A}, k+1, n}(x, y) \\ &= \int_{\mathcal{A}_k} \mathbb{K}^{\mathbb{A}, k}(x, z) \mathbb{K}_R^{\mathbb{A}, n+1}(z, y) \lambda(dz) - \int_{\mathcal{A}_{k+1}} \mathbb{K}^{\mathbb{A}, k+1}(x, w) \mathbb{K}_R^{\mathbb{A}, n}(w, y) \lambda(dw) \\ &= \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} \mathbb{K}^{\mathbb{A}, k}(x, z) \mathbb{K}_R(z, w) \mathbb{K}_R^{\mathbb{A}, n}(w, y) - \mathbb{K}^{\mathbb{A}, k}(x, z) \mathbb{K}(z, w) \mathbb{K}_R^{\mathbb{A}, n}(w, y) \lambda(dz) \lambda(dw) \\ &= \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} \mathbb{K}^{\mathbb{A}, k}(x, z) \mathbb{K}_R^{\mathbb{A}, n}(w, y) \left(\mathbb{K}_R(z, w) - \mathbb{K}(z, w) \right) \lambda(dz) \lambda(dw). \end{aligned}$$

By the Schwartz inequality for the last term, we have

$$\begin{aligned} & \left| \mathbb{L}_R^{\mathbb{A}, k, n+1}(x, y) - \mathbb{L}_R^{\mathbb{A}, k+1, n}(x, y) \right|^2 \\ & \leq \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} |\mathbb{K}^{\mathbb{A}, k}(x, z) \mathbb{K}_R^{\mathbb{A}, n}(w, y)|^2 \lambda(dz) \lambda(dw) \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} |\mathbb{K}_R(z, w) - \mathbb{K}(z, w)|^2 \lambda(dz) \lambda(dw). \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\mathcal{A}_0 \times \mathcal{A}_m} \left| \mathbb{L}_R^{\mathbb{A}, k, n+1}(x, y) - \mathbb{L}_R^{\mathbb{A}, k+1, n}(x, y) \right|^2 \lambda(dx) \lambda(dy) \\ & \leq \int_{\mathcal{A}_0 \times \mathcal{A}_k} |\mathbb{K}^{\mathbb{A}, k}(x, z)|^2 \lambda(dx) \lambda(dz) \int_{\mathcal{A}_{k+1} \times \mathcal{A}_m} |\mathbb{K}_R^{\mathbb{A}, n}(w, y)|^2 \lambda(dw) \lambda(dy) \\ & \quad \times \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} |\mathbb{K}_R(z, w) - \mathbb{K}(z, w)|^2 \lambda(dz) \lambda(dw). \end{aligned} \tag{2.3.15}$$

Recall that $k+n = m-1$. Then $0 \leq n \leq m-1$. Let $\mathbb{A}' = \mathcal{A}_{k+1} \times \dots \times \mathcal{A}_m$ and (k', n') be such that $k' + n' = n$. Then by replacing m by n in (2.3.14) we have

$$\sup_R \int_{\mathcal{A}_{k+1} \times \mathcal{A}_m} |\mathbb{L}_R^{\mathbb{A}', k', n'}(w, y)|^2 \lambda(dw) \lambda(dy) < \infty. \tag{2.3.16}$$

Take $(k', n') = (0, n)$. Then $\mathbb{L}_R^{\mathbb{A}', 0, n}(x, y) = \mathbb{K}_R^{\mathbb{A}, n}(x, y)$ by (2.3.12). Hence from (2.3.16)

$$\sup_R \int_{\mathcal{A}_{k+1} \times \mathcal{A}_m} |\mathbb{K}_R^{\mathbb{A}, n}(w, y)|^2 \lambda(dw) \lambda(dy) < \infty.$$

From this, (2.3.11), and Lemma 2.3.1 (2), the last term in (2.3.15) goes to zero as $R \rightarrow \infty$. Therefore, we see that

$$\begin{aligned} & \left(\int_{\mathcal{A}_0 \times \mathcal{A}_m} \left| \mathbb{L}_R^{\mathbb{A},k,n+1}(x,y) - \mathbb{K}^{\mathbb{A},m}(x,y) \right|^2 \lambda(dx) \lambda(dy) \right)^{\frac{1}{2}} \\ & \leq \sum_{p=0}^n \left(\int_{\mathcal{A}_0 \times \mathcal{A}_m} \left| \mathbb{L}_R^{\mathbb{A},k+p,n+1-p}(x,y) - \mathbb{L}_R^{\mathbb{A},k+p+1,n-p}(x,y) \right|^2 \lambda(dx) \lambda(dy) \right)^{\frac{1}{2}} \\ & \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

Hence (2.3.13) holds for m .

We deduce (2.3.14) for m from (2.3.13) for m immediately.

We now apply (2.3.14) to obtain (3). Let $\sigma = (1, 2, \dots, m)$.

$$\begin{aligned} & \int_{\mathbb{A}} \left\{ \prod_{p=1}^m \mathbb{K}_R(x_p, x_{\sigma(p)}) - \prod_{p=1}^m \mathbb{K}(x_p, x_{\sigma(p)}) \right\} \lambda^m(d\mathbf{x}) \\ & = \sum_{k=0}^{m-1} \int_{\mathcal{A}_m} \left\{ \mathbb{L}_R^{\mathbb{A},k,m-k}(x,x) - \mathbb{L}_R^{\mathbb{A},k+1,m-k-1}(x,x) \right\} \lambda(dx). \end{aligned} \quad (2.3.17)$$

Let $k+n=m$ and $n \geq 1$. Then

$$\begin{aligned} & \int_{\mathcal{A}_m} \left\{ \mathbb{L}_R^{\mathbb{A},k,n}(x,x) - \mathbb{L}_R^{\mathbb{A},k+1,n-1}(x,x) \right\} \lambda(dx) \\ & = \int_{\mathcal{A}_m} \left(\int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} \left\{ \mathbb{K}^{\mathbb{A},k}(x,z) \mathbb{K}_R(z,w) \mathbb{K}_R^{\mathbb{A},n-1}(w,x) \right. \right. \\ & \quad \left. \left. - \mathbb{K}^{\mathbb{A},k}(x,z) \mathbb{K}(z,w) \mathbb{K}_R^{\mathbb{A},n-1}(w,x) \right\} \lambda(dz) \lambda(dw) \right) \lambda(dx) \\ & = \int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} \int_{\mathcal{A}_m} \mathbb{K}^{\mathbb{A},k}(x,z) \mathbb{K}_R^{\mathbb{A},n-1}(w,x) \lambda(dx) \left(\mathbb{K}_R(z,w) - \mathbb{K}(z,w) \right) \lambda(dz) \lambda(dw). \end{aligned}$$

By the Schwarz inequality,

$$\begin{aligned} & \left| \int_{\mathcal{A}_m} \left\{ \mathbb{L}_R^{\mathbb{A},k,n}(x,x) - \mathbb{L}_R^{\mathbb{A},k+1,n-1}(x,x) \right\} \lambda(dx) \right| \\ & \leq \left(\int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} \left| \int_{\mathcal{A}_m} \mathbb{K}^{\mathbb{A},k}(x,z) \mathbb{K}_R^{\mathbb{A},n-1}(w,x) \lambda(dx) \right|^2 \lambda(dz) \lambda(dw) \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\mathcal{A}_k \times \mathcal{A}_{k+1}} |\mathbb{K}_R(z,w) - \mathbb{K}(z,w)|^2 \lambda(dz) \lambda(dw) \right)^{\frac{1}{2}}. \end{aligned}$$

Recall that $k+n=m$. Then $k+n-1=m-1$. From (2.3.14) for $m-1$ and Lemma 2.3.1 (2), the last term goes to zero as $R \rightarrow \infty$. This combined with (2.3.17) implies (2.3.6). $\square$

Let $\lambda_{\mathbb{I}(\ell)}$ be the counting measure on $\mathbb{I}(\ell)$ as before. We can regard $\mathbf{K}_{\mathbb{F}(\ell)}$ as an operator on $L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$ such that $\mathbf{K}_{\mathbb{F}(\ell)}p(i) = \sum_{j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j)p(j)$. We now prove that the $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ -determinantal point process $\nu_{\mathbb{F}(\ell)}$ exists.

Lemma 2.3.2. *Let $\text{Spec}(\mathbf{K}_{\mathbb{F}(\ell)})$ be the spectrum of $\mathbf{K}_{\mathbb{F}(\ell)}$. Then*

$$\text{Spec}(\mathbf{K}_{\mathbb{F}(\ell)}) \subset [0, \infty). \quad (2.3.18)$$

If $\alpha < 0$,

$$\text{Spec}(\mathbf{K}_{\mathbb{F}(\ell)}) \subset [0, -\frac{1}{\alpha}]. \quad (2.3.19)$$

In particular, there exists a unique, α -determinantal point process $\nu_{\mathbb{F}(\ell)}$ on $\mathbb{I}(\ell)$ associated with $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$.

Proof. Recall that $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ is an orthonormal basis of $L^2(S, \lambda)$. Let $U : L^2(S, \lambda) \rightarrow L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$ be the unitary operator such that $U(f_{\ell,i}) = e_{\ell,i}$, where $\{e_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ is the canonical orthonormal basis of $L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$. Then by Lemma 2.3.1 we see that $\mathbf{K}_{\mathbb{F}(\ell)} = U\mathbf{K}U^{-1}$. Hence $\mathbf{K}_{\mathbb{F}(\ell)}$ and $\mathbf{K}$ have the same spectrum. We thus obtain (2.3.18) and (2.3.19) from (A1). Because $\mathbf{K}_{\mathbb{F}(\ell)}$ is Hermitian symmetric, the second claim is clear from (2.3.18), (2.3.19), (A1), and Theorem 1.2 of [26]. $\square$

Lemma 2.3.3. *Let $\mathcal{B}_{\ell,i} = \text{supp}(f_{\ell,i})$ be as in (2.2.18). Then, for $i, j \in \mathbb{I}(\ell)$ and $\mathcal{A} \in \mathfrak{F}_{\ell}$,*

$$\int_{\mathcal{A}} f_{\ell,i}(x)f_{\ell,j}(x)\lambda(dx) = \begin{cases} 1 & (i = j, \mathcal{B}_{\ell,i} \subset \mathcal{A}) \\ 0 & (\text{otherwise}) \end{cases}. \quad (2.3.20)$$

Proof. We recall that $\mathcal{B}_{\ell,i}$ is the support of $f_{\ell,i}$ by (2.2.18). Suppose $i = j$ and $\mathcal{B}_{\ell,i} \subset \mathcal{A}$. Then from (2.3.1) we obtain

$$\int_{\mathcal{A}} f_{\ell,i}(x)f_{\ell,j}(x)\lambda(dx) = \int_S f_{\ell,i}(x)f_{\ell,i}(x)\lambda(dx) = 1. \quad (2.3.21)$$

Suppose that $i = j$ and that $\mathcal{B}_{\ell,i} \not\subset \mathcal{A}$. Then, using $\mathcal{A} \in \mathfrak{F}_{\ell}$, (2.2.5), and (2.2.19), we deduce that $\mathcal{B}_{\ell,i} \cap \mathcal{A} = \emptyset$. Because $\mathcal{B}_{\ell,i} = \text{supp}(f_{\ell,i})$, we obtain

$$\int_{\mathcal{A}} f_{\ell,i}(x)f_{\ell,j}(x)\lambda(dx) = 0. \quad (2.3.22)$$

Finally, suppose $i \neq j$. Because $\mathcal{A} \in \mathfrak{F}_{\ell}$, we see that $\mathcal{B}_{\ell,i} \subset \mathcal{A}$ or $\mathcal{B}_{\ell,i} \cap \mathcal{A} = \emptyset$. The same also holds for $\mathcal{B}_{\ell,j}$. In any case, we obtain (2.3.22) from (2.3.2). From (2.3.21) and (2.3.22), we obtain (2.3.20). $\square$

Proof of Theorem 2.2.1. Let $\mathbb{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ as in Theorem 2.2.1. Then, because $\mathcal{A}_n \in \Delta(\ell)$ for all $n = 1, \dots, m$, we deduce from (2.2.24), (2.2.2), and (2.1.1) that

$$\int_{\mathbb{A}} \rho_{\mathfrak{S}_\ell}^m(\mathbf{x}) \lambda^m(d\mathbf{x}) = \int_{\mathbb{A}} \det_\alpha[\mathbf{K}(x_p, x_q)]_{p,q=1}^m \lambda^m(d\mathbf{x}), \quad (2.3.23)$$

where $\mathbf{x} = (x_1, \dots, x_m)$. From a straightforward calculation and Lemma 2.3.1, we obtain

$$\begin{aligned} & \int_{\mathbb{A}} \det_\alpha[\mathbf{K}(x_p, x_q)]_{p,q=1}^m \lambda^m(d\mathbf{x}) \\ &= \int_{\mathbb{A}} \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \prod_{p=1}^m \mathbf{K}(x_p, x_{\sigma(p)}) \lambda^m(d\mathbf{x}) \\ &= \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \int_{\mathbb{A}} \prod_{p=1}^m \mathbf{K}(x_p, x_{\sigma(p)}) \lambda^m(d\mathbf{x}) \\ &= \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \lim_{R \rightarrow \infty} \int_{\mathbb{A}} \prod_{p=1}^m \mathbf{K}_R(x_p, x_{\sigma(p)}) \lambda^m(d\mathbf{x}), \end{aligned} \quad (2.3.24)$$

where $\mathbf{K}_R$ is defined by (2.3.5). We note that $\cup_{i=1}^m \mathcal{A}_i$ is relatively compact. Hence the last line in (2.3.24) follows from Lemma 2.3.1 (3).

$$\begin{aligned} & \int_{\mathbb{A}} \prod_{p=1}^m \mathbf{K}_R(x_p, x_{\sigma(p)}) \lambda^m(d\mathbf{x}) \\ &= \int_{\mathbb{A}} \prod_{p=1}^m \left(\sum_{i_p \in \mathbb{I}(\ell; R)} \mathbf{K}_{\mathbb{F}(\ell)}(i_p, j_p) f_{\ell, i_p}(x_p) f_{\ell, j_p}(x_{\sigma(p)}) \right) \lambda^m(d\mathbf{x}) \\ &= \int_{\mathbb{A}} \left(\sum_{\mathbf{i}, \mathbf{j} \in \mathbb{I}(\ell; R)^m} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, j_p) f_{\ell, i_p}(x_p) f_{\ell, j_p}(x_{\sigma(p)}) \right) \lambda^m(d\mathbf{x}) =: J(R) \end{aligned} \quad (2.3.25)$$

Here, $\mathbf{i} = (i_1, \dots, i_m)$, $\mathbf{j} = (j_1, \dots, j_m) \in \mathbb{I}(\ell)^m$. From Lemma 2.3.3,

$$\begin{aligned} J(R) &= \int_{\mathbb{A}} \left(\sum_{\mathbf{i}, \mathbf{j} \in \mathbb{I}(\ell; R)^m} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, j_p) f_{\ell, i_p}(x_p) f_{\ell, j_{\sigma^{-1}(p)}}(x_p) \right) \lambda^m(d\mathbf{x}) \\ &= \int_{\mathbb{A}} \left(\sum_{\mathbf{i} \in \mathbb{I}(\ell; R)^m \cap \mathbb{I}(\ell)(\mathbb{A})} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_{\sigma(p)}) |f_{\ell, i_p}(x_p)|^2 \right) \lambda^m(d\mathbf{x}) \\ &= \sum_{\mathbf{i} \in \mathbb{I}(\ell; R)^m \cap \mathbb{I}_\ell(\mathbb{A})} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_{\sigma(p)}) \rightarrow \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_{\sigma(p)}) \quad \text{as } R \rightarrow \infty. \end{aligned} \quad (2.3.26)$$

The convergence in the last line follows from Lemma 2.3.1 (2) and the Schwarz inequality. Multiplying $\alpha^{m-\nu(\sigma)}$ and summing over $\sigma \in \mathfrak{S}_m$ in the last term, we see that

$$\begin{aligned}
& \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_{\sigma(p)}) \\
&= \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \sum_{\sigma \in \mathfrak{S}_m} \alpha^{m-\nu(\sigma)} \prod_{p=1}^m \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_{\sigma(p)}) \\
&= \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \det_\alpha [\mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_q)]_{p,q=1}^m \\
&= \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}).
\end{aligned} \tag{2.3.27}$$

Combining (2.3.23)–(2.3.27) we deduce (2.2.26), which completes the proof. $\square$

2.4 Proof of Theorem 2.2.2 and Theorem 2.2.3

2.4.1 Proof of Theorem 2.2.2

Let ϱ^m be the m -point correlation function of $(\nu_{\mathbb{F}(\ell)} \diamond \lambda_{f_\ell}) \circ \mathbf{u}_\ell^{-1}|_{\mathfrak{S}_\ell}$. Then it suffices for (2.2.32) to prove

$$\rho_{\mathfrak{S}_\ell}^m(\mathbf{x}) = \varrho^m(\mathbf{x}). \tag{2.4.1}$$

From (2.1.6) and $\mathfrak{F}_\ell = \sigma[\mathcal{A}_{\ell,i}; i \in I(\ell)]$, we see that $\rho_{\mathfrak{S}_\ell}^m$ and ϱ^m are $\mathfrak{F}_\ell^m$ -measurable. Let $m = m_1 + \cdots + m_k$. Let $\mathbb{A} = \mathcal{A}_1^{m_1} \times \cdots \times \mathcal{A}_k^{m_k} \in \Delta(\ell)^m$ such that $\mathcal{A}_p \cap \mathcal{A}_q = \emptyset$ if $p \neq q$. Let $\mathbf{i} = (i_n)_{n=1}^m = (\mathbf{i}_1, \dots, \mathbf{i}_k) \in \mathbb{I}(\ell)^m$ such that $\mathbf{i}_n \in \mathbb{I}(\ell)^{m_n}$. From Theorem 2.2.1, we see that

$$\int_{\mathbb{A}} \rho_{\mathfrak{S}_\ell}^m(\mathbf{x}) \lambda^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}). \tag{2.4.2}$$

By the definition of correlation functions, (2.2.30), and (2.2.31), we see that

$$\begin{aligned}
\sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}) &= \int_{\mathbb{I}(\ell)} \prod_{n=1}^k \frac{\mathbf{i}(\mathbb{I}_\ell(\mathcal{A}_n))!}{(\mathbf{i}(\mathbb{I}_\ell(\mathcal{A}_n)) - m_n)!} \nu_{\mathbb{F}(\ell)}(d\mathbf{i}) \\
&= \int_{\mathcal{S}} \prod_{n=1}^k \frac{\mathbf{s}(\mathcal{A}_n)!}{(\mathbf{s}(\mathcal{A}_n) - m_n)!} (\nu_{\mathbb{F}(\ell)} \diamond \lambda_{f_\ell}) \circ \mathbf{u}_\ell^{-1}|_{\mathfrak{S}_\ell}(d\mathbf{s}) \\
&= \int_{\mathbb{A}} \varrho^m(\mathbf{x}) \lambda^m(d\mathbf{x}).
\end{aligned} \tag{2.4.3}$$

Combining (2.4.2) and (2.4.3), we deduce that

$$\int_{\mathbb{A}} \rho_{\mathfrak{G}_\ell}^m(\mathbf{x}) \lambda^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}) = \int_{\mathbb{A}} \varrho^m(\mathbf{x}) \lambda^m(d\mathbf{x}). \quad (2.4.4)$$

From (2.4.4), we obtain (2.4.1). This completes the proof of Theorem 2.2.2. $\square$

2.4.2 Proof of Theorem 2.2.3

Let $\mathbf{A} \in \mathfrak{G}_\ell$. From Theorem 2.2.2 and regular conditional probability of $\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}$ with respect to $\sigma[\iota_\ell]$, we see that

$$\begin{aligned} \mu|_{\mathfrak{G}_\ell}(\mathbf{A}) &= (\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}) \circ \mathbf{u}_\ell^{-1}|_{\mathfrak{G}_\ell}(\mathbf{A}) \\ &= \int_{\mathbb{I}(\ell)} (\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}) \circ \iota_\ell^{-1}(di) \nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}(\mathbf{u}_\ell^{-1}(\mathbf{A}) | \iota_\ell(\omega) = i) \\ &= \int_{\mathbb{I}(\ell)} (\nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}) \circ \iota_\ell^{-1}(di) \nu_{\mathbb{F}(\ell)} \diamond \lambda_{\mathbb{F}(\ell)}(\kappa_{\ell,i}^{-1} \circ \mathbf{u}_{\ell,i}^{-1}(\mathbf{A}) | \iota_\ell(\omega) = i) \\ &= \int_{\mathbb{I}(\ell)} \nu_{\mathbb{F}(\ell)}(di) \lambda_{f_{\ell,i}} \circ \mathbf{u}_{\ell,i}^{-1}(\mathbf{A}) \\ &= \int_{\mathbb{I}(\ell)} \nu_{\mathbb{F}(\ell)}(di) \lambda_{f_{\ell,i}}^u(\mathbf{A}). \end{aligned} \quad (2.4.5)$$

Here the forth line in (2.4.5) follows from the fact $\mathbf{u}_\ell(\omega) = \mathbf{u}_{\ell,i}(\kappa_{\ell,i}(\omega))$ for each $\omega \in \underline{\Omega}(\ell)$ with $\iota_\ell(\omega) = i$. From (2.4.5), we obtain Theorem 2.2.3. $\square$

2.5 Tail triviality of determinantal point processes

Let S be a locally compact Hausdorff space with countable basis with metric $\mathbf{d}$. Fix a point $o \in S$ as the origin. Set $S_r = \{x \in S; \mathbf{d}(o, x) < r\}$. Assume that each S_r is relatively compact. Note that this notion depends on the choice of metric $\mathbf{d}$ on S .

Denote $\mathbf{S}$ by the configuration space over S . For a Borel set A , we denote by $\pi_A : \mathbf{S} \rightarrow \mathbf{S}$ the projection of configuration such that $\mathbf{s}(\cdot) \mapsto \mathbf{s}(\cdot \cap A)$. Denote by $\text{Tail}(\mathbf{S})$ the tail σ -field such that

$$\text{Tail}(\mathbf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}].$$

Note that $\text{Tail}(\mathbf{S})$ is determined independently of the choice of $\mathbf{d}$.

Theorem 2.5.1. *Assume (A1) and (A2). Let μ be the $(\mathbf{K}, \mathbf{m})$ -determinantal point process on S . Then μ is tail trivial. That is, $\mu(A) \in \{0, 1\}$ for all $A \in \text{Tail}(\mathbf{S})$.*

Proof. Let $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$ be partitions of S satisfying (A2). Define new partitions $\Delta(\ell; r) = \{\mathcal{A}_{\ell,i}; i \in I_r(\ell)\}$ by

$$\Delta(\ell; r) = \{\mathcal{A}_{\ell,i}; \mathcal{A}_{\ell,i} \cap S_r = \emptyset\}.$$

Then for fixed $r \in \mathbb{N}$, $\Delta(\ell; r) \prec \Delta(\ell+1; r)$ for each $\ell \in \mathbb{N}$. Let $\mathfrak{G}(\ell, r)$ be sub- σ -fields on $\mathbf{S}$ by $\Delta(\ell; r)$ such that

$$\mathfrak{G}(\ell, r) = \sigma[\{\mathbf{s}(\mathcal{A}) = m\}; \mathcal{A} \in \Delta(\ell, r), m \in \mathbb{N}].$$

Then for each r , $\mathfrak{G}(\ell, r)$ is increasing in ℓ and we have

$$\sigma[\pi_{S_r^c}] = \sigma\left[\bigcup_{\ell \in \mathbb{N}} \mathfrak{G}(\ell, r)\right].$$

Hence for any $A \in \sigma[\pi_r^c]$, martingale convergence theorem implies that

$$\lim_{\ell \rightarrow \infty} \mu(A | \mathfrak{G}(\ell, r)) = 1_A \text{ in } L^1(\mathbf{S}, \mu). \quad (2.5.1)$$

Let $A \in \text{Tail}(\mathbf{S})$. Then from (2.5.1) we can take an increasing sequence $\{\ell_r\}_{r \in \mathbb{N}}$ such that

$$\lim_{r \rightarrow \infty} \mu(A | \mathfrak{G}(\ell_r, r)) = 1_A \text{ in } L^1(\mathbf{S}, \mu).$$

Hence

$$A \in \overline{\bigcap_{q \in \mathbb{N}} \sigma\left[\bigcup_{r \geq q} \mathfrak{G}(\ell_r, r)\right]}. \quad (2.5.2)$$

Let $\tilde{\Delta}(t) = \{\tilde{\mathcal{A}}_{k,i}\}_{i \in \mathbb{N}}$ be partitions of S generated by $\Delta(\ell_r, r)$ such that $r \geq t$. Then $\tilde{\Delta}(t) \succ \tilde{\Delta}(t+1)$ for each $t \in \mathbb{N}$. Define decreasing sub- σ -fields on $\mathbf{S}$ by $\tilde{\Delta}(t)$ such that

$$\mathfrak{H}(t) = \sigma[\{\mathbf{s}(\tilde{\mathcal{A}}) = m\}; \tilde{\mathcal{A}} \in \tilde{\Delta}(t), m \in \mathbb{N}].$$

Then $\mathfrak{H}(t)$ is decreasing in $t \in \mathbb{N}$. By (2.5.2), we have $A \in \mathfrak{H}(t)$ for each $t \in \mathbb{N}$. Hence backward martingale theorem implies that

$$\lim_{t \rightarrow \infty} \mu(A | \mathfrak{H}(t)) = 1_A \text{ in } L^1(\mathbf{S}, \mu). \quad (2.5.3)$$

For each $t \in \mathbb{N}$, set $D_t^c \subset S$ such that

$$D_t^c = S \setminus \bigcup_{\mathcal{A} \in \Delta(\ell_t, t)} \mathcal{A}.$$

Let $\tilde{\Delta}(k; t)$ be partitions on S generated by $\Delta(l_t + k; t)$. Then $\{\tilde{\Delta}(k; t); k \in \mathbb{N}\}$ satisfies (A2) as $\mathbf{m}$ -partitions on D_t^c . Let μ_t be the restriction of μ on $\sigma[\pi_{D_t^c}]$. Denote by $\mathbb{I}(t)$ the collection of indexes of partitions $\{\tilde{\Delta}(k; t); k \in \mathbb{N}\}$ defined in (??). Let $\mathbb{F}(t)$ be the associated orthonormal basis satisfying (??)–(??). Then by Lemma 2 in Part I, we can define $(\mathbf{K}_{\mathbb{F}(t)}, \lambda_{\mathbb{I}(t)})$ -determinantal point process on $\nu_{\mathbb{F}(t)}$. Denote by $\mathbb{I}(t)$ the configuration space over $\mathbb{I}(t)$. Let $\Pi_t : \mathbb{I}(t) \rightarrow \pi_{S^c}(\mathbf{S})$ be a projection that sends each atom of $\omega = \sum_n \delta_{i_n} \in \mathbf{Conf}(\mathbb{I}(t))$ to the center of the support of the orthonormal function f_{i_n} . By definition, $\tilde{\Delta}(t) \prec \tilde{\Delta}(k; t)$ for each $k \in \mathbb{N}$. Hence Theorem 2.2 in Part I implies that for $B \in \mathfrak{H}(t)$

$$\nu_{\mathbb{F}(t)} \circ \Pi_t(B) = \mu_t(B).$$

Because $\Pi_t^{-1}(\bigcap_{s \geq t} \mathfrak{H}(s)) \subset \text{Tail}(\mathbb{I}(t))$, Theorem 7.15 in [9] implies that

$$\mu_t(A) \in \{0, 1\}.$$

From this together with (2.5.3), we obtain the claim. □

Chapter 3

Bernoulli property of determinantal point processes

We prove the Bernoulli property for determinantal point processes on $\mathbb{R}^d$ with translation-invariant kernels. For the determinantal point processes on $\mathbb{Z}^d$ with translation-invariant kernels, the Bernoulli property was proved by Lyons and Steif [11] and Shirai and Takahashi [27]. As its continuum version, we prove an isomorphism between the translation-invariant determinantal point processes on $\mathbb{R}^d$ with translation-invariant kernels and homogeneous Poisson point processes. For this purpose, we also prove the Bernoulli property for the tree representations of the determinantal point processes.

3.1 Main statement: Bernoulli property of determinantal point processes

We consider an isomorphism problem of measure-preserving dynamical systems among translation-invariant point processes on $\mathbb{R}^d$ such as the homogeneous Poisson point processes and the determinantal point processes with translation-invariant kernel functions.

The homogeneous Poisson point process is a point process in which numbers of particles on disjoint subsets obey independently Poisson distributions. It is parameterized using intensity $r > 0$. From the general theory of Ornstein and Weiss [17], homogeneous Poisson point processes are isomorphic to each other regardless of the value of r .

The determinantal point process is a point process for which the determinants of its kernel function give its correlation functions. It describes a repulsive particle system and appears in various mathematical systems such as uniform spanning trees,

the zeros of a hyperbolic Gaussian analytic function with a Bergman kernel, and the eigenvalue distribution of random matrices.

These two classes of point processes have different properties in correlations among particles. For example, determinantal point processes have negative associations [9]. The sine point process is a typical example of a translation-invariant determinantal point process that has number rigidity [5]. In contrast, Poisson point processes do not have this property because the particles are regionally independent. Nevertheless, we prove they are isomorphic to each other.

We start by recalling the isomorphism theory.

An automorphism S of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a bi-measurable bijection such that $\mathbb{P} \circ S^{-1} = \mathbb{P}$. Let $S_G = \{S_g : g \in G\}$ be a group of automorphisms of $(\Omega, \mathcal{F}, \mathbb{P})$ parametrized by a group G . A measure-preserving dynamical system of G -action is the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$. We call $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ the G -action system for short.

Let $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ and $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ be G -action systems. A factor map is a measurable map $\phi : \Omega \rightarrow \Omega'$ such that

$$\mathbb{P} \circ \phi^{-1} = \mathbb{P}', \quad \phi \circ S_g(x) = S'_g \circ \phi(x) \text{ for each } g \in G \text{ and a.s. } x \in \Omega.$$

In this case, we call $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ the ϕ -factor of $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ or simply a factor of $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$. An isomorphism is a bi-measurable bijection $\phi : \Omega \rightarrow \Omega'$ such that both ϕ and ϕ^{-1} are factor maps. If there exists an isomorphism $\phi : \Omega \rightarrow \Omega'$, then $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ and $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ are said to be isomorphic.

Let $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ be a G -action system with a measurable map ϕ from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$. Then $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ is a G -action system. Here, $(\Omega', \mathcal{F}', \mathbb{P}')$ is the completion of $(\Omega', \sigma[\phi], \mathbb{P} \circ \phi^{-1})$, and $S'_G = \{\phi \circ S_g \circ \phi^{-1} : g \in G\}$. We also call the G -action system $(\Omega', \mathcal{F}', \mathbb{P}', S'_G)$ the ϕ -factor of $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$.

A typical system with a discrete group action is a Bernoulli shift. A G -action Bernoulli shift is a system formed from the direct product of a probability space over G and the canonical shift. Ornstein [14, 15] proved that the $\mathbb{Z}$ -action Bernoulli shifts with equal entropy are isomorphic to each other. We call a system $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ Bernoulli if $(\Omega, \mathcal{F}, \mathbb{P}, S_G)$ is isomorphic to a Bernoulli shift. Ornstein and Weiss [17] extended the isomorphism theory to amenable group actions. As a consequence of the general theory, all the homogeneous Poisson point processes on $\mathbb{R}^d$ are isomorphic to each other regardless of their intensity.

Let X be a locally compact Hausdorff space with countable basis. We denote by $\mathbf{Conf}(\mathbb{R}^d)$ the set of all nonnegative integer-valued Radon measures on X . We equip $\mathbf{Conf}(\mathbb{R}^d)$ with the vague topology, under which $\mathbf{Conf}(\mathbb{R}^d)$ is a Polish space. We call a Borel probability measure μ on $\mathbf{Conf}(\mathbb{R}^d)$ a point process on X . We say μ is simple when $\xi(\{x\}) \in \{0, 1\}$ for each $x \in X$ for a.s. $\xi \in \mathbf{Conf}(\mathbb{R}^d)$.

Let μ be a point process on X . Throughout this paper, we write the completion of μ by the same symbol. We also write $(\mathbf{Conf}(\mathbb{R}^d), \mu, T_G)$ as the G -action

system made of the completion of $(\text{Conf}(\mathbb{R}^d), \mathcal{B}(\text{Conf}(\mathbb{R}^d)), \mu)$ and a G -action group of automorphisms T_G .

A homogeneous Poisson point process with intensity $r > 0$ is the point process on $\mathbb{R}^d$ satisfying:

- (1) $\xi(A)$ has a Poisson distribution with mean $r|A|$ for each $A \in \mathcal{B}(\mathbb{R}^d)$.
 - (2) $\xi(A_1), \dots, \xi(A_k)$ are independent for any disjoint subsets $A_1, \dots, A_k \in \mathcal{B}(\mathbb{R}^d)$.
- Here, $\xi(A)$ is the number of particles on A for $\xi \in \text{Conf}(\mathbb{R}^d)$ and $|A|$ is the Lebesgue measure of A .

A determinantal point process μ on X is a point process associated with a kernel function $\mathsf{K} : X \times X \rightarrow \mathbb{C}$ and a Radon measure λ on X , for which the n -point correlation function with respect to λ is given by

$$\rho_n(x_1, \dots, x_n) = \det[\mathsf{K}(x_i, x_j)]_{i,j=1}^n \quad (3.1.1)$$

for each $n \in \mathbb{N}$. See Definition 3.4.1 for the definition of the n -point correlation function. We call μ a (K, λ) -determinantal point process. If the context is clear, we omit λ calling μ a K -determinantal point process. Throughout this paper, we assume that λ is the Lebesgue measure if $X = \mathbb{R}^d$.

Now, we state the main theorem:

Theorem 3.1.1. *Let $\hat{K} \in L^1(\mathbb{R}^d)$ such that $\hat{K}(t) \in [0, 1]$ for a.e. $t \in \mathbb{R}^d$. Let μ^K be a determinantal point process on $\mathbb{R}^d$ with translation-invariant kernel K such that*

$$K(x, y) = \int_{\mathbb{R}^d} \hat{K}(t) e^{2\pi i(x-y) \cdot t} dt. \quad (3.1.2)$$

Then $(\text{Conf}(\mathbb{R}^d), \mu^K, \mathsf{T}_{\mathbb{R}^d})$ is isomorphic to a Poisson point process. Here, $\mathsf{T}_a : \sum_i \delta_{x_i} \mapsto \sum_i \delta_{x_i+a}$ for $a \in \mathbb{R}^d$ and $\mathsf{T}_{\mathbb{R}^d} = \{\mathsf{T}_a : a \in \mathbb{R}^d\}$.

We remark that the assumption for K in Theorem 3.1.1 implies the following condition (1)–(4) with $X = \mathbb{R}^d$ and the Lebesgue measure λ .

- (1) $K : X \times X \rightarrow \mathbb{C}$ is Hermitian symmetric.
- (2) For each compact set $A \subset X$, the integral operator K on $L^2(A, \lambda)$ is of trace class.
- (3) $\text{Spec} K \subset [0, 1]$.
- (4) $K(x, y) = \mathsf{K}(x - y, 0)$.

Under assumptions (1)–(3), there exists a unique (K, λ) -determinantal point process μ with the kernel function K [24, 26].

The K -determinantal point process μ satisfying (1)–(4) above is translation invariant because its n -correlation functions are translation invariant.

For determinantal point processes on $\mathbb{Z}^d$ with translation-invariant kernel and the counting measure, Lyons and Steif [11] and Shirai and Takahashi [27] independently proved the Bernoulli property, the latter giving a sufficient condition for the weak

Bernoulli property under the assumption $K : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$ satisfying (1), (2), $\text{Spec}(K) \subset (0, 1)$, and (4). We recall that the weak Bernoulli property is stronger than the Bernoulli property. Lyons and Steif [11] proved the Bernoulli property for the case K satisfying (1)–(4). Theorem 3.1.1 is its continuum version.

One of the ideas in [11] is using the dbar distance, which is a metric on the set of $\mathbb{Z}^d$ -action systems; the Bernoulli property is closed under this metric [16, 17, 28]. However, the dbar distance does not work for systems with infinite entropy because entropy is continuous with respect to the dbar distance. In general, a translation-invariant point process on $\mathbb{R}^d$ has infinite entropy. Therefore, we cannot apply the dbar distance to our case. Therefore, we construct point processes on a discrete set that approximate the determinantal point process on $\mathbb{R}^d$. We prove the Bernoulli property of the discrete point processes. In turn, we can prove the isomorphism of the determinantal point process on $\mathbb{R}^d$ and the Poisson point process via the tree representation [20].

To prove Theorem 3.1.1, we apply the general theory given by Ornstein and Weiss [17]. We quote them in the form applicable to the $\mathbb{R}^d$ - and $\mathbb{Z}^d$ -actions. We also refer to [16] for the $\mathbb{Z}$ - and $\mathbb{R}$ -actions, and [28] for the $\mathbb{Z}^d$ -action.

The outline of this paper is as follows. In Section 3.2, we recall notions related to the Bernoulli property. In Section 3.3, we introduce the kernel functions that approximate the determinantal kernel K in Theorem 3.1.1 uniformly on any compact set on $\mathbb{R}^d$. In Section 3.4, we introduce the tree representations of the determinantal point processes on $\mathbb{R}^d$. We combine these representations with the kernels introduced in Section 3.3. The tree representations are determinantal point processes on $\mathbb{Z}^d \times \mathbb{N}$ and are translation invariant with respect to the first coordinate. In Section 3.5, we prove the Bernoulli property of the tree representation using the properties of the dbar distance introduced in Section 3.2. In Section 3.6, we prove Theorem 3.1.1 using the Bernoulli property of the tree representations.

3.2 Notions related to the Bernoulli property

In this section, we collect properties of point processes without determinantal structure and notions related to the Bernoulli property.

We first recall the notion of monotone coupling. For $\zeta^i = \{\zeta_z^i\}_{z \in \mathbb{Z}^d} \in \{0, 1\}^{\mathbb{Z}^d}$ ($i = 1, 2$), we write $\zeta^1 \leq \zeta^2$ if $\zeta_z^1 \leq \zeta_z^2$ for each $z \in \mathbb{Z}^d$. We equip $\{0, 1\}^{\mathbb{Z}^d}$ with the product topology. We call a continuous function $f : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ a monotone function on $\{0, 1\}^{\mathbb{Z}^d}$ if $\zeta^1 \leq \zeta^2$ implies $f(\zeta^1) \leq f(\zeta^2)$. Let $\mathcal{B}$ be the Borel σ -field of $\{0, 1\}^{\mathbb{Z}^d}$. For probability measures μ and ν on $(\{0, 1\}^{\mathbb{Z}^d}, \mathcal{B})$, we write $\mu \leq \nu$ if for

each monotone function f ,

$$\int_{\{0,1\}^{\mathbb{Z}^d}} f d\mu \leq \int_{\{0,1\}^{\mathbb{Z}^d}} f d\nu.$$

Let ν_1 and ν_2 be probability measures on $\{0,1\}^{\mathbb{Z}^d}$. We say a probability measure γ on $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ is a monotone coupling of ν_1 and ν_2 if the following hold:

- (1) $\gamma(A \times \{0,1\}^{\mathbb{Z}^d}) = \nu_1(A)$ for $A \in \mathcal{B}$.
- (2) $\gamma(\{0,1\}^{\mathbb{Z}^d} \times B) = \nu_2(B)$ for $B \in \mathcal{B}$.
- (3) $\gamma(\{(\zeta^1, \zeta^2) \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}; \zeta^1 \leq \zeta^2\}) = 1$.

Lemma 3.2.1 (e.g. [8]). *For probability measures μ and ν on $\{0,1\}^{\mathbb{Z}^d}$, the following statements are equivalent:*

- (1) $\mu \leq \nu$.
- (2) *There exists a monotone coupling of μ and ν .*

We naturally regard a simple point process μ on $\mathbb{Z}^d \times \mathbb{N}$ as a probability measure on $\{0,1\}^{\mathbb{Z}^d \times \mathbb{N}}$, denoted by the same symbol μ . We write $\mu \leq \nu$ for simple point processes μ and ν if the corresponding probability measures on $\{0,1\}^{\mathbb{Z}^d \times \mathbb{N}}$ satisfy $\mu \leq \nu$. We introduce the notion of monotone coupling for simple point processes on $\mathbb{Z}^d \times \mathbb{N}$ from that of the corresponding probability measures on $\{0,1\}^{\mathbb{Z}^d \times \mathbb{N}}$ in an obvious fashion.

Fix $N \in \mathbb{N}$. We set $[N] = \{1, \dots, N\}$. Let $Q^N = \{Q_{z,l}^N : (z, l) \in \mathbb{Z}^d \times [N]\}$ be a partition of $\mathbb{Z}^d \times \mathbb{N}$ such that

$$Q_{z,l}^N = \begin{cases} \{(z, l)\} & \text{for } l \in [N-1] \\ \{(z, m) \in \mathbb{Z}^d \times \mathbb{N}; m \geq l\} & \text{for } l = N \end{cases} \quad (3.2.1)$$

for each $(z, l) \in \mathbb{Z}^d \times [N]$. For $\xi \in \mathbf{Conf}(\mathbb{Z}^d \times \mathbb{N})$, we set

$$\omega_{z,l}^N(\xi) = 1_{\{\xi(Q_{z,l}^N) \geq 1\}}.$$

Let $\varpi_N : \mathbf{Conf}(\mathbb{Z}^d \times \mathbb{N}) \rightarrow \{0,1\}^{\mathbb{Z}^d \times [N]}$ denote the map

$$\xi \mapsto \{\omega_{z,l}^N(\xi)\}_{(z,l) \in \mathbb{Z}^d \times [N]}. \quad (3.2.2)$$

We denote the image measure $\nu \circ \varpi_N^{-1}$ by ν_N for a point process ν on $\mathbb{Z}^d \times \mathbb{N}$.

Proposition 3.2.2. *Let μ and ν be simple point processes on $\mathbb{Z}^d \times \mathbb{N}$. Assume $\mu \leq \nu$. Then $\mu_N \leq \nu_N$.*

Proof. By assumption and Lemma 3.2.1, there exists a monotone coupling γ of μ and ν . Let $\gamma_N(\xi, \eta) = \gamma \circ (\varpi_N(\xi), \varpi_N(\eta))^{-1}$. Then for $A \in \mathcal{B}(\{0, 1\}^{\mathbb{Z}^d \times [N]})$,

$$\begin{aligned} \gamma_N(A \times \{0, 1\}^{\mathbb{Z}^d \times [N]}) &= \gamma(\{(\xi, \eta); (\varpi_N(\xi), \varpi_N(\eta)) \in A \times \{0, 1\}^{\mathbb{Z}^d \times [N]}\}) \\ &= \gamma(\varpi_N^{-1}(A) \times \mathbf{Conf}(\mathbb{Z}^d \times \mathbb{N})) \\ &= \mu(\varpi_N^{-1}(A)) \\ &= \mu_N(A). \end{aligned}$$

The third equation follows from the fact that γ is a coupling of μ and ν . Because the same is true for $\{0, 1\}^{\mathbb{Z}^d \times [N]} \times A$, we find

$$\gamma_N(\{0, 1\}^{\mathbb{Z}^d \times [N]} \times A) = \nu_N(A).$$

Moreover, by $\gamma_N(\xi, \eta) = \gamma \circ (\varpi_N(\xi), \varpi_N(\eta))^{-1}$

$$\begin{aligned} &\gamma_N(\{(\zeta, \omega) \in \{0, 1\}^{\mathbb{Z}^d \times [N]} \times \{0, 1\}^{\mathbb{Z}^d \times [N]}; \zeta \leq \omega\}) \\ &= \gamma(\{(\xi, \eta) \in \mathbf{Conf}(\mathbb{Z}^d \times \mathbb{N}) \times \mathbf{Conf}(\mathbb{Z}^d \times \mathbb{N}); \varpi_N(\xi) \leq \varpi_N(\eta)\}) \\ &= 1. \end{aligned}$$

The last equation follows from the fact that γ is a monotone coupling of μ and ν . Hence γ_N is a monotone coupling of μ_N and ν_N . From this and Lemma 3.2.1, we prove the claim. $\square$

We recall the notion of being finitely dependent, which is a sufficient condition for the Bernoulli property. See, e.g., [11].

Definition 3.2.3. *Let Ω be a countable set.*

(1) *A probability measure ν on $\Omega^{\mathbb{Z}^d}$ is called r -dependent if, for each $R, S \subset \mathbb{Z}^d$,*

$$\inf\{\mathbf{d}(z, w); z \in R, w \in S\} \geq r \Rightarrow \sigma[\pi_R] \text{ and } \sigma[\pi_S] \text{ are independent.}$$

Here, $\mathbf{d}(z, w)$ is the graph distance on $\mathbb{Z}^d$ and $\pi_R : \Omega^{\mathbb{Z}^d} \rightarrow \Omega^R$ is the projection given by $\{\omega_z\}_{z \in \mathbb{Z}^d} \mapsto \{\omega_z\}_{z \in R}$.

(2) *ν is called finitely dependent if ν is r -dependent for some $r \in \mathbb{N}$.*

Let $\mathcal{P}_{inv}(M)$ be the set of translation-invariant probability measures on $[M]^{\mathbb{Z}^d}$. For $x, y \in \mathbb{Z}^d$, define $x < y$ if $x_i < y_i$ for $i = \min\{j = 1, \dots, d; x_j \neq y_j\}$. For $P, Q \subset \mathbb{Z}^d$, we set $P < Q$ if $x < y$ for all $x \in P$ and $y \in Q$.

Definition 3.2.4 (Very weak Bernoulli). *We call $\nu \in \mathcal{P}_{inv}(M)$ very weak Bernoulli if for each $\epsilon > 0$, there is a rectangle $R \subset \mathbb{Z}^d$ such that if, for any finite set $Q = \{x_1, \dots, x_m\} < R$, there exists an $\mathcal{A} \subset \sigma[\pi_Q]$ satisfying (3.2.3) and (3.2.4):*

$$\nu\left(\bigcup_{A \in \mathcal{A}} A\right) > 1 - \epsilon. \quad (3.2.3)$$

$$\inf_{\gamma \in \Gamma(\nu|_R, \nu_A|_R)} \mathbb{E}^\gamma\left[\frac{1}{\#R} \sum_{z \in R} 1_{\{X_z \neq Y_z\}}\right] < \epsilon \text{ for } A \in \mathcal{A}. \quad (3.2.4)$$

Here ν_A denotes the conditional probability measure under A , $\nu|_R = \nu \circ \pi_R^{-1}$, and $\nu_A|_R = \nu_A \circ \pi_R^{-1}$. Furthermore, $\Gamma(\nu|_R, \nu_A|_R)$ is the collection of the couplings of $\nu|_R$ and $\nu_A|_R$, and $((X_z)_{z \in R}, (Y_z)_{z \in R}) \in [M]^R \times [M]^R$.

Lemma 3.2.5 (e.g. [11]). *If $\nu \in \mathcal{P}_{inv}(M)$ is finitely dependent, then ν is very weak Bernoulli.*

Proof. By definition, there exists a r_0 such that ν is r_0 -dependent. For $\epsilon > 0$, let $R \subset \mathbb{Z}^d$ be a rectangle such that $(r_0)^d / \#R < \epsilon$. Let $Q = \{z_1, \dots, z_m\} \subset \mathbb{Z}^d$ be a finite set such that $Q < R$. We set

$$Q_{r_0} = \{w \in \mathbb{Z}^d; \mathbf{d}(z, w) \leq r_0 \text{ for some } z \in Q\}.$$

Then $\#R \cap Q_{r_0} \leq (r_0)^d$. By r_0 -dependence, $\sigma[\pi_Q]$ and $\sigma[\pi_{Q_{r_0}^c}]$ are independent under ν . Hence for each $A \in \sigma[\pi_Q]$, $\nu = \nu_A$ on $\sigma[\pi_{Q_{r_0}^c}]$. Let γ_A be the coupling of $\nu \circ \pi_R^{-1}$ and $\nu_A \circ \pi_R^{-1}$ such that $X_z = Y_z$ for $z \in R \cap Q_{r_0}^c$ and $X_z \perp Y_z$ for $z \in R \cap Q_{r_0}$ under γ_A . Then

$$\mathbb{E}^{\gamma_A}\left[\frac{1}{\#R} \sum_{z \in R} 1_{\{X_z \neq Y_z\}}\right] \leq \frac{(r_0)^d}{\#R} < \epsilon.$$

This proves the claim. $\square$

The very weak Bernoulli property is equivalent to the Bernoulli property for elements of $\mathcal{P}_{inv}(M)$:

Lemma 3.2.6 ([16, 17, 28]). *For $\nu \in \mathcal{P}_{inv}(M)$, the following statements are equivalent:*

- (1) ν is very weak Bernoulli.
- (2) ν is isomorphic to a Bernoulli shift.

From Lemma 3.2.5 and Lemma 3.2.6, we obtain:

Proposition 3.2.7 (e.g. [11]). *If $\nu \in \mathcal{P}_{inv}(M)$ is finitely dependent, then ν is isomorphic to a Bernoulli shift.*

Let μ and $\nu \in \mathcal{P}_{inv}(M)$. Define $\bar{d} : \mathcal{P}_{inv}(M) \times \mathcal{P}_{inv}(M) \rightarrow [0, 1]$ by

$$\bar{d}(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \gamma(\{(\zeta, \omega) \in [M]^{\mathbb{Z}^d} \times [M]^{\mathbb{Z}^d}; \zeta_0 \neq \omega_0\}). \quad (3.2.5)$$

Then $\bar{d}$ gives a metric on $\mathcal{P}_{inv}(M)$. The Bernoulli property is closed under $\bar{d}$:

Lemma 3.2.8 ([16, 17, 28]). *Let ν and $\{\nu_n : n \in \mathbb{N}\}$ be elements of $\mathcal{P}_{inv}(M)$. Suppose that $\lim_{n \rightarrow \infty} \bar{d}(\nu_n, \nu) = 0$ and that each ν_n is isomorphic to a Bernoulli shift. Then ν is isomorphic to a Bernoulli shift.*

We quote Theorem 5 in III.6 in [17]:

Lemma 3.2.9 ([16, 17]). *Let $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{Z}^d})$ be an ergodic system. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be an increasing sequence of $S_{\mathbb{Z}^d}$ -invariant sub- σ -fields. Let $\bigvee_{n \in \mathbb{N}} \mathcal{F}_n$ be the completion of $\sigma[\bigcup_{n \in \mathbb{N}} \mathcal{F}_n]$. Assume that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ satisfies (3.2.6) and (3.2.7):*

$$\bigvee_{n \in \mathbb{N}} \mathcal{F}_n = \mathcal{F}. \quad (3.2.6)$$

$$\mathcal{F}_n\text{-factor is isomorphic to a Bernoulli shift for each } n. \quad (3.2.7)$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, S_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Proposition 3.2.10. *Let $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu, \mathbf{T}_{\mathbb{Z}^d})$ be ergodic. Let ν be simple. Suppose that there exists a sequence $\{\nu_r : r \in \mathbb{N}\}$ of point processes on $\mathbb{Z}^d \times \mathbb{N}$ such that*

$$\nu_{r,N} \text{ is isomorphic to a Bernoulli shift for each } r \text{ and } N \in \mathbb{N}, \quad (3.2.8)$$

$$\lim_{r \rightarrow \infty} \bar{d}(\nu_{r,N}, \nu_N) = 0 \text{ for each } N \in \mathbb{N}. \quad (3.2.9)$$

Here, $\nu_{r,N} = \nu_r \circ \varpi_N^{-1}$ and $\nu_N = \nu \circ \varpi_N^{-1}$. Then, ν is isomorphic to a Bernoulli shift.

Proof. Recall that $Q^N = \{Q_{z,l}^N : (z, l) \in \mathbb{Z}^d \times [N]\}$ is a partition of $\mathbb{Z}^d \times \mathbb{N}$. Here, $Q_{z,l}^N$ is defined in (3.2.1). Then Q^N becomes finer as $N \rightarrow \infty$ and $\bigvee_{N \in \mathbb{N}} Q^N$ separates points of $\mathbb{Z}^d \times \mathbb{N}$ by construction. Here, $\bigvee_{N \in \mathbb{N}} Q^N$ is the refinement of partitions $\{Q^N\}_{N \in \mathbb{N}}$. From this, we obtain that $\{\sigma[\varpi_N]\}_{N \in \mathbb{N}}$ is increasing and $\bigvee_{N \in \mathbb{N}} \sigma[\varpi_N]$ separates points of $\text{Conf}(\mathbb{Z}^d \times \mathbb{N})$. Hence $\{\sigma[\varpi_N]\}_{N \in \mathbb{N}}$ satisfies (3.2.6).

From the assumptions (3.2.8) and (3.2.9) and Lemma 3.2.8, ν_N is isomorphic to a Bernoulli shift. Hence $\{\sigma[\varpi_N]\}_{N \in \mathbb{N}}$ satisfies (3.2.7).

From the above and Lemma 3.2.9, the claim holds. $\square$

3.3 Approximations of the determinantal kernel

In this section, we introduce three approximations of the kernel K introduced in (3.1.2).

For $r > 0$, let $w_r : \mathbb{R}^d \rightarrow \mathbb{R}$ be the product of the tent function such that

$$w_r(x) = \prod_{j=1}^d (1 - |x_j|/r) 1_{\{|x_j| < r\}}(x).$$

We denote by $\hat{w}_r$ its Fourier transform

$$\hat{w}_r(t) = \int_{\mathbb{R}^d} w_r(x) e^{2\pi i x \cdot t} dx = r^{-d} \prod_{j=1}^d \left(\frac{\sin \pi r t_j}{\pi t_j} \right)^2.$$

Let $\hat{K} \in L^1(\mathbb{R}^d)$ such that $\hat{K}(t) \in [0, 1]$ for a.s. $t \in \mathbb{R}^d$. Set $\hat{K}_r(t) = \hat{K} * \hat{w}_r(t)$. Then $\hat{K}_r(t) \in [0, 1]$ for a.e. $t \in \mathbb{R}^d$. Let

$$\underline{K}_r(x, y) = \int_{\mathbb{R}^d} (\hat{K}_r(t) \wedge \hat{K}(t)) e^{2\pi i(x-y) \cdot t} dt, \quad (3.3.1)$$

$$K_r(x, y) = \int_{\mathbb{R}^d} \hat{K}_r(t) e^{2\pi i(x-y) \cdot t} dt, \quad (3.3.2)$$

$$\overline{K}_r(x, y) = \int_{\mathbb{R}^d} (\hat{K}_r(t) \vee \hat{K}(t)) e^{2\pi i(x-y) \cdot t} dt. \quad (3.3.3)$$

Here, $a \wedge b = \max\{a, b\}$ and $a \vee b = \min\{a, b\}$ for $a, b \in \mathbb{R}$, respectively. Then $\underline{K}_r$, K_r , and $\overline{K}_r$ satisfy (1)–(4) before Theorem 3.1.1.

For $K : X \times X \mapsto \mathbb{C}$, we denote $0 \leq K$ if K is nonnegative definite as an integral operator on $L^2(\mathbb{R}^d)$ and $K_1 \leq K_2$ if $K_2 - K_1$ is nonnegative definite.

Lemma 3.3.1. *Let $\underline{K}_r$, K_r , and $\overline{K}_r$ be as (3.3.1), (3.3.2), and (3.3.3), respectively. Then*

$$\underline{K}_r \leq K \leq \overline{K}_r, \quad (3.3.4)$$

$$\underline{K}_r \leq K_r \leq \overline{K}_r. \quad (3.3.5)$$

Proof. By construction, we see

$$\begin{aligned} \hat{K}_r(t) \wedge \hat{K}(t) &\leq \hat{K}(t) \leq \hat{K}_r(t) \vee \hat{K}(t) \\ \hat{K}_r(t) \wedge \hat{K}(t) &\leq \hat{K}_r(t) \leq \hat{K}_r(t) \vee \hat{K}(t). \end{aligned}$$

From (3.3.1)–(3.3.3) combined with the above inequalities, we obtain (3.3.4) and (3.3.5). $\square$

3.4 Tree representations of determinantal point processes

In this section, we introduce the tree representations of determinantal point processes on $\mathbb{R}^d$. Then we apply it to the determinantal point processes associated with the kernels introduced in Section 3.3. Before doing so, we recall the definition and well-known facts about determinantal point processes.

Let μ be a point process on X . A locally integrable symmetric function $\rho^n : X^n \rightarrow [0, \infty)$ is called the n -point correlation function of μ (with respect to a Radon measure λ on X) if

$$\mathbb{E}^\mu \left[\prod_{i=1}^k \frac{\xi(A_i)!}{(\xi(A_i) - n_i)!} \right] = \int_{A_1^{n_1} \times \dots \times A_k^{n_k}} \rho^n(x_1, \dots, x_n) \lambda(dx_1) \cdots \lambda(dx_n) \quad (3.4.1)$$

for any disjoint Borel subsets $A_1, \dots, A_k$ and for any $n_i \in \mathbb{N}$, $i = 1, \dots, k$ such that $\sum_{i=1}^k n_i = n$. Let $K : X \times X \rightarrow \mathbb{C}$. We call μ a determinantal point process with kernel K and Radon measure λ if the n -point correlation function ρ^n of μ with respect to λ satisfies (3.1.1) for each n .

Assume $K : X \times X \rightarrow \mathbb{C}$ satisfies:

$$\overline{K(x, y)} = K(y, x). \quad (3.4.2)$$

$$\text{Spec}(K) \subset [0, 1]. \quad (3.4.3)$$

$$K_A \text{ is trace class for any compact } A \subset X. \quad (3.4.4)$$

Here, K in (3.4.3) is an integral operator on $L^2(X, \lambda)$ such that $Kf(x) = \int_X K(x, y)\lambda(dy)$ and K_A in (3.4.4) is its restriction on $L^2(A, \lambda)$. Then there exists a unique determinantal point process on X with kernel function K .

Next, we introduce the tree representations of the determinantal point processes. Let μ^K be the determinantal point process on $\mathbb{R}^d$ with kernel function K satisfying (3.4.2)–(3.4.4). First, we introduce a partition of $\mathbb{R}^d$ and the associated orthonormal basis on $L^2(\mathbb{R}^d)$. Let $P = \{P_z : z \in \mathbb{Z}^d\}$ be a partition of $\mathbb{R}^d$ such that each P_z is relatively compact and

$$P_{z+w} = P_z + w \text{ for } z, w \in \mathbb{Z}^d.$$

Here, $A + x = \{a + x; a \in A\}$ for $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Let $\Phi = \Phi_P = \{\phi_{z,l}\}_{(z,l) \in \mathbb{Z}^d \times \mathbb{N}}$ be an orthonormal basis on $L^2(\mathbb{R}^d)$ such that $\text{supp} \phi_{z,l} \subset P_z$ and

$$\phi_{z+w,l}(x) = \phi_{z,l}(x - w). \quad (3.4.5)$$

For the kernel function K above, let $K^\Phi : (\mathbb{Z}^d \times \mathbb{N}) \times (\mathbb{Z}^d \times \mathbb{N}) \rightarrow \mathbb{C}$ such that

$$K^\Phi(z, l; w, m) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_{z,l}(x) K(x, y) \phi_{w,m}(y) dx dy. \quad (3.4.6)$$

Lemma 3.4.1. *Assume that K satisfies (3.4.2)–(3.4.4) with respect to $L^2(\mathbb{R}^d)$. Then K^Φ satisfies (3.4.2)–(3.4.4) with respect to the counting measure on $\mathbb{Z}^d \times \mathbb{N}$.*

Proof. By assumption and (3.4.6), K^Φ satisfies (3.4.2) and (3.4.4). (3.4.3) follows from Lemma 2 in p.430 of [20]. $\square$

From Lemma 3.4.1 and the general theory in [24, 26], there exists a determinantal point process $\nu^{K,\Phi}$ on $\mathbb{Z}^d \times \mathbb{N}$ associated with K^Φ . We call $\nu^{K,\Phi}$ the tree representation of μ^K with respect to Φ .

Lemma 3.4.2 ([20]). *Let $\pi : \text{Conf}(\mathbb{Z}^d \times \mathbb{N}) \rightarrow \text{Conf}(\mathbb{Z}^d)$ such that*

$$\eta \mapsto \pi(\eta) = \sum_{z \in \mathbb{Z}^d} \eta(\{z\} \times \mathbb{N}) \delta_z.$$

Then for $A \in \sigma[\{\xi \in \text{Conf}(\mathbb{Z}^d \times \mathbb{N}); \xi(P_z) = n\}; z \in \mathbb{Z}^d, n \in \mathbb{N}]$,

$$\nu^{K,\Phi} \circ \pi^{-1}(A) = \mu^K(A).$$

Proof. From Theorem 2 on p.427 of [20], we easily obtain the claim. $\square$

We apply the tree representations for the translation-invariant kernels on $\mathbb{R}^d$ introduces in Section 3.3.

Assume that K is given by (3.1.2). Then K is translation invariant. Hence by construction K^Φ is translation invariant with respect to the first coordinate $\mathbb{Z}^d$. From this we see that $\nu^{K,\Phi}$ is translation invariant with respect to the first coordinate.

Define $\underline{K}_r^\Phi$, K_r^Φ , and $\overline{K}_r^\Phi$ similarly as (3.4.6) with replacement of K with $\underline{K}_r$, K_r , and $\overline{K}_r$ in (3.3.1)–(3.3.3), respectively. By construction, $\underline{K}_r$, K_r , and $\overline{K}_r$ satisfies (3.4.2)–(3.4.4). Hence $\underline{K}_r^\Phi$, K_r^Φ , and $\overline{K}_r^\Phi$ satisfy (3.4.2)–(3.4.4) with respect to the counting measure on $\mathbb{Z}^d \times \mathbb{N}$ by Lemma 3.4.2. Furthermore, $\underline{K}_r^\Phi$, K_r^Φ , and $\overline{K}_r^\Phi$ are translation invariant with respect to the first coordinate $\mathbb{Z}^d$.

Let $\underline{\nu}_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, and $\overline{\nu}_r^{K,\Phi}$ be $\underline{K}_r^\Phi$ -, K_r^Φ - and $\overline{K}_r^\Phi$ -determinantal point process, respectively. We remark that a determinantal point process ν on $\mathbb{Z}^d$ has no multiple points with probability 1. Hence we can regard ν as a probability measure on $\{0, 1\}^{\mathbb{Z}^d}$. We quote:

Lemma 3.4.3 ([9]). *Let $K_i : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$ satisfying (3.4.2)–(3.4.4) ($i = 1, 2$). Assume that $K_1 \leq K_2$. Let ν^{K_1} and ν^{K_2} be the determinantal point processes with K_1 and K_2 , respectively. Then there exists a monotone coupling of ν^{K_1} and ν^{K_2} .*

Applying Lemma 3.4.3, we obtain the following:

Lemma 3.4.4. *Let $\underline{\nu}_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, and $\bar{\nu}_r^{K,\Phi}$ be determinantal point processes on $\mathbb{Z}^d \times \mathbb{N}$ as above. Then*

$$\underline{\nu}_r^{K,\Phi} \leq \nu_r^{K,\Phi} \leq \bar{\nu}_r^{K,\Phi}, \quad (3.4.7)$$

$$\underline{\nu}_r^{K,\Phi} \leq \nu_r^{K,\Phi} \leq \bar{\nu}_r^{K,\Phi}. \quad (3.4.8)$$

Proof. Recall that Φ is the orthonormal basis of $L^2(\mathbb{R}^d)$ given in (3.4.5). Let $U : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{Z}^d \times \mathbb{N})$ be the unitary operator such that $U(\phi_{z,n}) = e_{z,n}$, where $\{e_{z,n}\}_{(z,n) \in \mathbb{Z}^d \times \mathbb{N}}$ is the canonical orthonormal basis of $L^2(\mathbb{Z}^d \times \mathbb{N})$. Then by Lemma 1 in Section 3 of [20], we see that $K^\Phi = UKU^{-1}$. From this and Lemma 3.3.1, we obtain

$$\underline{K}_r^\Phi \leq K^\Phi \leq \bar{K}_r^\Phi, \quad (3.4.9)$$

$$\underline{K}_r^\Phi \leq K_r^\Phi \leq \bar{K}_r^\Phi. \quad (3.4.10)$$

From (3.4.9) and (3.4.10) combined with Lemma 3.4.3, we conclude (3.4.7) and (3.4.8). $\square$

Recall that $\underline{K}_r^\Phi$, K_r^Φ , and $\bar{K}_r^\Phi$ are translation invariant with respect to the first coordinate. Hence $\underline{\nu}_r^{K,\Phi}$, $\nu_r^{K,\Phi}$, and $\bar{\nu}_r^{K,\Phi}$ are also translation invariant with respect to the first coordinate. We regard $\mathbf{T}_{\mathbb{Z}^d} = \{\mathbf{T}_a : a \in \mathbb{Z}^d\}$ as a translation on $\text{Conf}(\mathbb{Z}^d \times \mathbb{N})$ such that

$$\mathbf{T}_a : \sum_i \delta_{(z_i, l_i)} \mapsto \sum_i \delta_{(z_i + a, l_i)} \text{ for } a \in \mathbb{Z}^d.$$

Then $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \underline{\nu}_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \bar{\nu}_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, and $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \bar{\nu}_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$ are $\mathbb{Z}^d$ -action systems.

3.5 Bernoulli property of tree representations

We continue the setting of Section 3.4. Let K^Φ be the kernel defined by (3.4.6). Let $\nu_r^{K,\Phi}$ be the K^Φ -determinantal point process as before. The purpose of this section is to prove the Bernoulli property for $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$.

Let ϖ_N be the map defined by (3.2.2). Let $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_{r,N}^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$ denote the ϖ_N -factor of $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$. Here, $\mathbf{T}_{\mathbb{Z}^d}$ in $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_{r,N}^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$ is the shift of $\{0, 1\}^{\mathbb{Z}^d \times [N]}$ such that for each $a \in \mathbb{Z}^d$

$$\mathbf{T}_a : \omega = \{\omega_{z,l}\}_{(z,l) \in \mathbb{Z}^d \times [N]} \mapsto \{\omega_{z+a,l}\}_{(z,l) \in \mathbb{Z}^d \times [N]}.$$

We also denote ϖ_N -factors of $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \underline{\nu}_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, and $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \bar{\nu}_r^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$ by $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \underline{\nu}_{r,N}^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_{r,N}^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, and $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \bar{\nu}_{r,N}^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$, respectively. We shall prove that $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_N^{K,\Phi}, \mathbf{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Lemma 3.5.1.

$$\begin{aligned}\underline{\nu}_{r,N}^{K,\Phi} &\leq \nu_N^{K,\Phi} \leq \overline{\nu}_{r,N}^{K,\Phi}, \\ \underline{\nu}_{r,N}^{K,\Phi} &\leq \nu_{r,N}^{K,\Phi} \leq \overline{\nu}_{r,N}^{K,\Phi}.\end{aligned}$$

Proof. From Proposition 3.2.2 and Lemma 3.4.4, we obtain the claim. $\square$

Lemma 3.5.2. $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_{r,N}^{K,\Phi}, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Proof. We identify $\{0, 1\}^{\mathbb{Z}^d \times [N]}$ with $[2^N]^{\mathbb{Z}^d}$ and $\nu_{r,N}^{K,\Phi}$ with an element of $\mathcal{P}_{inv}(2^N)$, respectively. We shall prove that $\nu_{r,N}^{K,\Phi}$ is finitely dependent. For this it only remains to prove that $\nu_r^{K,\Phi}$ is finitely dependent because $(\{0, 1\}^{\mathbb{Z}^d \times [N]}, \nu_{r,N}^{K,\Phi}, \mathbb{T}_{\mathbb{Z}^d})$ is the ϖ_N -factor of $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu_r^{K,\Phi}, \mathbb{T}_{\mathbb{Z}^d})$.

Let d be the graph distance as before. Let $r_0 > 0$ such that for each $z, w \in \mathbb{Z}^d$,

$$d(z, w) \geq r_0 \Rightarrow \inf\{|z_i - w_i|; i = 1, \dots, d\} \geq r.$$

For $P, Q \subset \mathbb{Z}^d \times \mathbb{N}$, we define a pseudo-distance by

$$d(P, Q) = \inf\{d(z, w); (z, l) \in P, (w, m) \in Q\}.$$

Let $P, Q \subset \mathbb{Z}^d \times \mathbb{N}$ be finite sets such that $d(P, Q) \geq r_0$. Then

$$K_r^\Phi(z, l; w, m) = 0 \quad \text{for } (z, l) \in P, (w, m) \in Q. \quad (3.5.1)$$

For $P \subset \mathbb{Z}^d \times \mathbb{N}$, we define a cylinder set by

$$1^P = \{\omega \in \text{Conf}(\mathbb{Z}^d \times \mathbb{N}); \omega(\{(z, l)\}) = 1 \text{ for all } (z, l) \in P\}.$$

By construction, $1^P \cap 1^Q = 1^{P \cup Q}$. Therefore

$$\begin{aligned}\nu_r^{K,\Phi}(1^P \cap 1^Q) &= \nu_r^{K,\Phi}(1^{P \cup Q}) \\ &= \det[K_r^\Phi(z, l; w, m)]_{(z,l),(w,m) \in P \cup Q} \\ &= \det[K_r^\Phi(z, l; w, m)]_{(z,l),(w,m) \in P} \det[K_r^\Phi(z, l; w, m)]_{(z,l),(w,m) \in Q} \\ &= \nu_r^{K,\Phi}(1^P) \nu_r^{K,\Phi}(1^Q).\end{aligned} \quad (3.5.2)$$

The third equality follows from (3.5.1).

Let $R, S \subset \mathbb{Z}^d$ such that $d(R \times \mathbb{N}, S \times \mathbb{N}) \geq r_0$. From (3.5.2) and the π - λ theorem,

$$\nu_r^{K,\Phi}(A \cap B) = \nu_r^{K,\Phi}(A) \nu_r^{K,\Phi}(B)$$

for each $A \in \sigma[\pi_{R \times \mathbb{N}}]$ and $B \in \sigma[\pi_{S \times \mathbb{N}}]$. Hence $\nu_{r,N}^{K,\Phi}$ is r_0 -dependent.

From this and Proposition 3.2.7, the claim holds. $\square$

Lemma 3.5.3. *For each N ,*

$$\lim_{r \rightarrow \infty} \bar{d}(\nu_N^{K,\Phi}, \nu_{r,N}^{K,\Phi}) = 0. \quad (3.5.3)$$

Proof. Because $\bar{d}$ is a metric on $\mathcal{P}_{inv}(M)$,

$$\bar{d}(\nu_N^{K,\Phi}, \nu_{r,N}^{K,\Phi}) \leq \bar{d}(\underline{\nu}_{r,N}^{K,\Phi}, \nu_N^{K,\Phi}) + \bar{d}(\underline{\nu}_{r,N}^{K,\Phi}, \nu_{r,N}^{K,\Phi}), \quad (3.5.4)$$

$$\bar{d}(\nu_N^{K,\Phi}, \nu_{r,N}^{K,\Phi}) \leq \bar{d}(\nu_N^{K,\Phi}, \bar{\nu}_{r,N}^{K,\Phi}) + \bar{d}(\nu_{r,N}^{K,\Phi}, \bar{\nu}_{r,N}^{K,\Phi}). \quad (3.5.5)$$

From Lemma 3.2.1 and Lemma 3.5.1, there exists a monotone coupling γ_N of $\underline{\nu}_{r,N}^{K,\Phi}$ and $\nu_N^{K,\Phi}$. By definition (3.2.5) of $\bar{d}$, we deduce

$$\begin{aligned} \bar{d}(\underline{\nu}_{r,N}^{K,\Phi}, \nu_N^{K,\Phi}) &\leq \gamma_N(\{(\omega_1, \omega_2) ; \omega_1(\{0\} \times \{l\}) \neq \omega_2(\{0\} \times \{l\}) \text{ for } \exists l \in [N]\}) \\ &\leq \sum_{l \in [N]} \gamma_N(\{(\omega_1, \omega_2) ; \omega_1(\{0\} \times \{l\}) \neq \omega_2(\{0\} \times \{l\})\}) \\ &= \sum_{l \in [N]} \{\nu_N^{K,\Phi}(\omega_1(\{0\} \times \{l\}) = 1) - \underline{\nu}_{r,N}^{K,\Phi}(\omega_2(\{0\} \times \{l\}) = 1)\}. \end{aligned} \quad (3.5.6)$$

The last equation follows from the fact that γ_N is a monotone coupling of $\underline{\nu}_{r,N}^{K,\Phi}$ and $\nu_N^{K,\Phi}$. Because of Lemma 3.5.1, (3.5.6) is true for $(\underline{\nu}_{r,N}^{K,\Phi}, \nu_{r,N}^{K,\Phi})$, $(\nu_N^{K,\Phi}, \bar{\nu}_{r,N}^{K,\Phi})$, and $(\nu_{r,N}^{K,\Phi}, \bar{\nu}_{r,N}^{K,\Phi})$. From this combined with (3.5.4) and (3.5.5), we obtain

$$\begin{aligned} \bar{d}(\nu_N^{K,\Phi}, \nu_{r,N}^{K,\Phi}) &\leq \sum_{l \in [N]} \{\bar{\nu}_{r,N}^{K,\Phi}(\omega_1(\{0\} \times \{l\}) = 1) - \underline{\nu}_{r,N}^{K,\Phi}(\omega_2(\{0\} \times \{l\}) = 1)\} \\ &= \sum_{l \in [N-1]} \{\bar{\nu}_r^{K,\Phi}(\omega_1(\{0\} \times \{l\}) = 1) - \underline{\nu}_r^{K,\Phi}(\omega_2(\{0\} \times \{l\}) = 1)\} \\ &\quad + \bar{\nu}_r^{K,\Phi}(\omega_1(\{0\} \times \mathbb{N} \setminus [N]) \geq 1) - \underline{\nu}_r^{K,\Phi}(\omega_2(\{0\} \times \mathbb{N} \setminus [N]) \geq 1). \end{aligned} \quad (3.5.7)$$

The last equation follows from the definitions of $\bar{\nu}_r^{K,\Phi}$ and $\underline{\nu}_r^{K,\Phi}$.

For (z, l) and (w, m) ,

$$|\bar{K}_r^\Phi(z, l; w, m) - \underline{K}_r^\Phi(z, l; w, m)| \quad (3.5.8)$$

$$\begin{aligned} &= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \{\bar{K}_r(x, y) - \underline{K}_r(x, y)\} \phi_{z,l}(x) \phi_{w,m}(y) dx dy \right| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |\bar{K}_r(x, y) - \underline{K}_r(x, y)| |\phi_{z,l}(x) \phi_{w,m}(y)| dx dy \\ &= \int_{\text{supp } \phi_{z,l} \times \text{supp } \phi_{w,m}} |\bar{K}_r(x, y) - \underline{K}_r(x, y)| |\phi_{z,l}(x) \phi_{w,m}(y)| dx dy. \end{aligned} \quad (3.5.9)$$

Because $\overline{K}_r, \underline{K}_r \in L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\phi_{z,l}$ and $\phi_{w,m}$ are orthonormal bases on $L^2(\mathbb{R}^d)$ with relatively compact support, the Schwarz inequality implies that

$$(3.5.9) \leq \left(\int_{\text{supp}\phi_{z,l} \times \text{supp}\phi_{w,m}} |\overline{K}_r(x, y) - \underline{K}_r(x, y)|^2 dx dy \right)^{\frac{1}{2}} \quad (3.5.10)$$

Because $\hat{K}_r \rightarrow \hat{K}$ in $L^1(\mathbb{R}^d)$ as $r \rightarrow \infty$, $\underline{K}_r$ and $\overline{K}_r$ converge to K uniformly on any compact set. Hence RHS of (3.5.10) goes to 0 as $r \rightarrow \infty$. This implies that (3.5.8) goes to 0 as $r \rightarrow \infty$. Hence for each compact set $R \subset \mathbb{Z}^d \times \mathbb{N}$,

$$\max \left\{ |\overline{K}_r^\Phi(z, l; w, m) - \underline{K}_r^\Phi(z, l; w, m)| ; (z, l), (w, m) \in R \right\} \rightarrow 0 \text{ as } r \rightarrow \infty.$$

From this and Proposition 3.10 in [26],

$$\overline{\nu}_r^{K, \Phi}, \underline{\nu}_r^{K, \Phi} \rightarrow \nu^{K, \Phi} \text{ weakly as } r \rightarrow \infty. \quad (3.5.11)$$

Finally, (3.5.7) and (3.5.11) imply (3.5.3). $\square$

Theorem 3.5.4. $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K, \Phi}, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Proof. From Proposition 3.2.10, Lemma 3.5.2 and Lemma 3.5.3, the claim holds. $\square$

3.6 Proof of Theorem 3.1.1

The purpose of this section is to complete the proof of Theorem 3.1.1.

We quote a general fact of isomorphism theory:

Lemma 3.6.1 ([16, 17]). *Let $(\Omega', \mathcal{F}', \mathbb{P}', \mathbb{S}'_{\mathbb{Z}^d})$ be a factor of $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{S}_{\mathbb{Z}^d})$. If $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{S}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift, then $(\Omega', \mathcal{F}', \mathbb{P}', \mathbb{S}'_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.*

For $n \in \mathbb{N}$, let $P_n = \{P_{n,z} : z \in \mathbb{Z}^d\}$ be a partition of $\mathbb{R}^d$ such that

$$P_{n,z} = \prod_{i=1}^d \left[\frac{z_i}{2^{n-1}}, \frac{z_i + 1}{2^{n-1}} \right), \quad z = (z_1, \dots, z_d) \in \mathbb{Z}^d.$$

Let $\Pi_{P_n} : \text{Conf}(\mathbb{R}^d) \rightarrow \text{Conf}(\mathbb{Z}^d)$ such that

$$\xi \mapsto \sum_{z \in \mathbb{Z}^d} \xi(P_{n,z}) \delta_z.$$

Then $\Pi_{P_n} \circ \mathbb{T}_z(\xi) = \mathbb{T}_z \circ \Pi_{P_n}(\xi)$ for each $z \in \mathbb{Z}^d$ and $\xi \in \text{Conf}(\mathbb{R}^d)$. Let $\mu_{P_n}^K = \mu^K \circ \Pi_{P_n}^{-1}$. Then $(\text{Conf}(\mathbb{Z}^d), \mu_{P_n}^K, \mathbb{T}_{\mathbb{Z}^d})$ is the Π_{P_n} -factor of $(\text{Conf}(\mathbb{R}^d), \mu^K, \mathbb{T}_{\mathbb{Z}^d})$.

Lemma 3.6.2. $(\text{Conf}(\mathbb{Z}^d), \mu_{P_n}^K, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Proof. Let $\Phi_n = \{\phi_{z,l}^n\}_{(z,l) \in \mathbb{Z}^d \times \mathbb{N}}$ be an orthonormal basis on $L^2(\mathbb{R}^d)$ such that $\phi_{z+w,l}^n(x) = \phi_{z,l}^n(x-w)$ and $\text{supp } \phi_{z,l}^n = P_{n,z}$. Let $\nu^{K,\Phi}$ be the tree representation of μ^K with respect to Φ_n . Let $\pi : \text{Conf}(\mathbb{Z}^d \times \mathbb{N}) \rightarrow \text{Conf}(\mathbb{Z}^d)$ such that

$$\eta \mapsto \pi(\eta) = \sum_{z \in \mathbb{Z}^d} \eta(\{z\} \times \mathbb{N}) \delta_z.$$

By construction, $\pi \circ \mathbb{T}_z(\eta) = \mathbb{T}_z \circ \pi(\eta)$ for each $z \in \mathbb{Z}^d$ and $\eta \in \text{Conf}(\mathbb{Z}^d \times \mathbb{N})$. From Lemma 3.4.2,

$$\nu^{K,\Phi} \circ \pi^{-1} = \mu_{P_n}^K.$$

Hence $(\text{Conf}(\mathbb{Z}^d), \mu_{P_n}^K, \mathbb{T}_{\mathbb{Z}^d})$ is the π -factor of $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathbb{T}_{\mathbb{Z}^d})$. From Theorem 3.5.4, $(\text{Conf}(\mathbb{Z}^d \times \mathbb{N}), \nu^{K,\Phi}, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift. From this and Lemma 3.6.1, the claim holds. $\square$

Lemma 3.6.3. $(\text{Conf}(\mathbb{R}^d), \mu^K, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift.

Proof. By construction, the sequence of partitions $\{P_n : n \in \mathbb{N}\}$ is increasingly finer and separates the points of $\mathbb{R}^d$. From this, we obtain that $\{\sigma[\Pi_{P_n}]\}_{n \in \mathbb{N}}$ is increasing and $\bigvee_{n \in \mathbb{N}} \sigma[\Pi_{P_n}]$ separates the points of $\text{Conf}(\mathbb{R}^d)$. Putting this result together with Lemma 3.6.2 and Lemma 3.2.9 implies the claim. $\square$

We quote Theorem 10 of III.6. in [17]:

Lemma 3.6.4 ([17]). *For an $\mathbb{R}^d$ -action system $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{S}_{\mathbb{R}^d})$, let $\mathbb{S}_{\mathbb{Z}^d} = \{\mathbb{S}_g : g \in \mathbb{Z}^d\}$ be the limitation on $\mathbb{Z}^d$ -action of $\mathbb{S}_{\mathbb{R}^d}$. If $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{S}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift with infinite entropy, then $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{S}_{\mathbb{R}^d})$ is isomorphic to a homogeneous Poisson point process.*

We are now ready to complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. From Lemma 3.6.3, $(\text{Conf}(\mathbb{R}^d), \mu^K, \mathbb{T}_{\mathbb{Z}^d})$ is isomorphic to a Bernoulli shift. Because the restriction of μ^K on $[0,1]^d$ is a non-atomic probability measure, the entropy of $(\text{Conf}(\mathbb{R}^d), \mu^K, \mathbb{T}_{\mathbb{Z}^d})$ is infinite. Putting this and Lemma 3.6.4 together implies the claim. $\square$

Chapter 4

Logarithmic derivative and Gibbs property

We prove that the existence of logarithmic derivatives of point processes on $\mathbb{R}^d$ implies their Gibbs property. As its application, we prove that determinantal point processes on $\mathbb{R}$ related to random matrices have continuous density. This implies that the Dirichlet form associated with the point processes becomes closable.

4.1 Gibbs property

Let μ be a point process on $\mathbb{R}^d$. Let $\mu_{R,m,\eta}$ be a regular conditional probability given by

$$\mu_{R,m,\eta}(d\xi) = \mu(\pi_{B_R}(\cdot) \in d\xi \mid \xi(B_R) = m, \pi_{B_R^c}\xi = \pi_{B_R^c}\eta).$$

Here, $B_R \subset \mathbb{R}^d$ is the open ball of radius R centered at the origin and π_A is the projection of configuration on $A \subset \mathbb{R}^d$ such that $\xi(\cdot) \mapsto \xi(\cdot \cap A)$. Denote by Λ the Poisson point process with intensity 1. Set $\Lambda_{R,m}(d\xi) = \Lambda(\pi_R(\cdot) \in d\xi \mid \xi(B_R) = m)$. In this paper, we say μ is Gibbsian if, for each $R, m \in \mathbb{N}$ and μ -a.s. η , $\mu_{R,m,\eta}$ is absolutely continuous with respect to $\Lambda_{R,m}$. This formulation seems weaker than that due to the DLR equations.

A conventional definition of the canonical Gibbs measure is given by the Dobrushin-Lanford-Ruelle equation (4.1.1) (cf. [22, 23]). Let $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$. For $\xi = \sum_i \delta_{x_i}, \eta = \sum_j \delta_{y_j} \in \text{Conf}(\mathbb{R}^d)$ and $R \in \mathbb{N}$, let

$$\mathcal{H}_{R,\eta}(\xi) = \sum_{x_i \in B_R} \Psi(x_i) + \sum_{i < j, x_i, x_j \in B_R} \Psi(x_i, x_j) + \sum_{x_i \in B_R, y_j \in B_R^c} \Psi(x_i, y_j).$$

Let $\Lambda_{R,m}$ be a conditional probability given by

$$\Lambda_{R,m}(d\xi) = \Lambda(\pi_R(\cdot) \in d\xi \mid \xi(B_R) = m).$$

We say μ is a canonical Gibbs measure for a free potential Φ and an interaction potential Ψ if μ satisfies the Dobrushin-Lanford-Ruelle equation

$$\mu_{R,m,\eta}(d\xi) = \frac{1}{Z} \exp(-\mathcal{H}_{R,\eta}(\xi)) \Lambda_{R,m}(d\xi) \quad (4.1.1)$$

for each $R, m \in \mathbb{N}$ and μ -a.s. η . By replacing equality by inequality in (4.1.1), the quasi-Gibbs measure is introduced in [19].

As above, our formulation of the Gibbs measure is weaker than the canonical Gibbs measure and the quasi Gibbs measure. However, due to remarks in Georgii-Yoo [6], existence of the Papangelou intensity is said to be Gibbsian in a general sense. On the other hand, for Gibbsian point processes, the continuity of Radon-Nikodym densities gives a sufficient condition for the closability of associated symmetric forms. For these reasons, it is essential to examine the weak Gibbs property and the continuity of the densities. We shall prove Gibbs measure (in the weak sense) is still useful for the construction of the dynamics if it admits the logarithmic derivative.

The logarithmic derivative is defined as the derivative of the reduced Campbell measure in the sense of distribution in the spatial direction. In this paper, we prove that the Gibbs property follows from the existence of the logarithmic derivative. Especially in the case $d = 1$, the density becomes continuous.

We apply this to a wide class of determinantal point processes on $\mathbb{R}^d$ introduced in [1]. See also [3, 4]. In particular, their kernels include 1 as their spectrum. For determinantal point processes on discrete sets of which spectrum does not contain 1, Shirai-Takahashi [27] established the Gibbs property in the DLR equations sense. In $\mathbb{R}^d$ case, Yoo [31] proved the Gibbs property for determinantal point processes with translation-invariant kernels of which spectrum does not contain 1. We prove the weaker Gibbs property for determinantal point processes on $\mathbb{R}$ with kernels that admit division property [1, 3, 4]. The spectrums of the kernels contain 1.

The organization of this paper is as follows. In Section 4.2, we introduce the Gibbs property and the logarithmic derivative and formulate our main results. Section 4.3 and Section 4.4 are devoted to the proofs. In Section 4.5, we give an application to a class of determinantal point processes on $\mathbb{R}$ with kernels admits division property. See Assumption 2.2. for the division property.

4.2 Main statement: Logarithmic derivative and Gibbs property

Consider the space of nonnegative integer valued Radon measures

$$\mathbf{Conf}(\mathbb{R}^d) := \{\xi = \sum_i \delta_{x_i} \mid \{x_i\} \text{ has no limit point in } \mathbb{R}^d\}.$$

We equip $\mathbf{Conf}(\mathbb{R}^d)$ with the vague topology. $\mathbf{Conf}(\mathbb{R}^d)$ equipped with the Borel σ -field is called a configuration space. A Borel probability measure μ on $\mathbf{Conf}(\mathbb{R}^d)$ is called a point process on $\mathbb{R}^d$.

For $A \subset \mathbb{R}^d$, we denote by $\pi_A : \mathbf{Conf}(\mathbb{R}^d) \rightarrow \mathbf{Conf}(\mathbb{R}^d)$ the projection of configurations such that $\xi(\cdot) \mapsto \xi(\cdot \cap A)$. Let $B_R \subset \mathbb{R}^d$ be the open ball of radius R centered at the origin. We set $\pi_R = \pi_{B_R}$ and $\pi_R^c = \pi_{B_R^c}$, respectively. We write $\xi_R^c = \pi_R^c(\xi)$. For $R \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\eta \in \mathbf{Conf}(\mathbb{R}^d)$, we set the regular conditional probability by

$$\mu_{R,m,\eta}(d\xi) = \mu(\pi_R(\cdot) \in d\xi \mid \xi(B_R) = m, \xi_R^c = \eta_R^c). \quad (4.2.1)$$

We denote by Λ the homogeneous Poisson point process on $\mathbb{R}^d$ with intensity 1. We set $\Lambda_{R,m}(d\xi) = \Lambda(\pi_R(\cdot) \in d\xi \mid \xi(B_R) = m)$.

Definition 4.2.1. A point process μ is called *Gibbsian* if for μ -a.s. η , $R \in \mathbb{N}$ and $m \in \mathbb{N}_0$, $\mu_{R,m,\eta}$ is absolutely continuous with respect to $\Lambda_{R,m}$.

Let μ be a point process on $\mathbb{R}^d$. A locally integrable symmetric function $\rho^n : X^n \rightarrow [0, \infty)$ is called the n -point correlation function of μ (with respect to the Lebesgue measure) if

$$\mathbb{E}^\mu \left[\prod_{i=1}^k \frac{\xi(A_i)!}{(\xi(A_i) - n_i)!} \right] = \int_{A_1^{n_1} \times \dots \times A_k^{n_k}} \rho^n(x_1, \dots, x_n) dx_1 \cdots dx_n$$

for any disjoint Borel subsets $A_1, \dots, A_k$ and for any $n_i \in \mathbb{N}$, $i = 1, \dots, k$ such that $\sum_{i=1}^k n_i = n$.

For $\xi \in \mathbf{Conf}(\mathbb{R}^d)$ and $m \in \mathbb{N}$, the factorial measure $\xi^{[m]}$ on $(\mathbb{R}^d)^m$ is defined by

$$\xi^{[m]}(dx_1 \cdots dx_m) = \xi(dx_1)(\xi - \delta_{x_1})(dx_2) \cdots (\xi - \sum_{n=1}^{m-1} \delta_{x_n})(dx_m). \quad (4.2.2)$$

The m -reduced Campbell measure $\mathcal{C}_\mu^{[m]}$ of a point process μ is a σ -finite Borel measure on $(\mathbb{R}^d)^m \times \mathbf{Conf}(\mathbb{R}^d)$ given by

$$\mathcal{C}_\mu^{[m]}(A \times \mathcal{A}) = \mathbb{E}^\mu \left[\int_A 1_{\mathcal{A}}(\xi - \sum_{n=1}^m \delta_{x_n}) \xi^{[m]}(dx_1 \cdots dx_m) \right]$$

for $A \in \mathcal{B}((\mathbb{R}^d)^m)$ and $\mathcal{A} \in \mathcal{B}(\text{Conf}(\mathbb{R}^d))$.

For $R \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, set

$$\text{Conf}(\mathbb{R}^d)_{R,m} = \{\xi \in \text{Conf}(\mathbb{R}^d); \xi(B_R) = m\}.$$

Let $\mathbf{l}_{R,m} : \text{Conf}(\mathbb{R}^d)_{R,m} \rightarrow (B_R)^m$ be a map such that

$$\mathbf{l}_{R,m}(\xi) = (\mathbf{l}_{R,m}^1(\xi), \mathbf{l}_{R,m}^2(\xi), \dots, \mathbf{l}_{R,m}^m(\xi))$$

and $\xi_R = \sum_{n=1}^m \delta_{\mathbf{l}_{R,m}^n(\xi)}$.

A function $\phi : \text{Conf}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is called local if there exists a compact set $K \subset \mathbb{R}^d$ such that ϕ is $\sigma[\pi_K]$ -measurable. For a local function ϕ such that $\sigma[\pi_R]$ -measurable, we define symmetric functions $\phi_{R,m} : (B_R)^m \rightarrow \mathbb{R}$ by the relation

$$\phi_{R,m}(\mathbf{l}_{R,m}(\xi)) = \phi(\xi), \quad \xi \in \text{Conf}(\mathbb{R}^d)_{R,m}. \quad (4.2.3)$$

Remark that $\phi_{R,m}$ is unique and $\phi(\xi) = \sum_{m=0}^{\infty} \phi_{R,m}(\mathbf{l}_{R,m}(\xi))$. Furthermore, $\phi_{R,m}$ is independent of the choice of R such that ϕ is $\sigma[\pi_R]$ -measurable.

A local function ϕ is said to be smooth if $\phi_{R,m}$ is smooth for each $R > Q$ and $m \in \mathbb{N}$. Here, Q is a positive number such that ϕ is $\sigma[\pi_Q]$ -measurable. Clearly, ϕ is smooth if $\phi_{R,m}$ is smooth for *some* $R > Q$ and each $m \in \mathbb{N}$.

Let $\mathcal{D}_o$ denote the space of all bounded local smooth functions on $\text{Conf}(\mathbb{R}^d)$.

Definition 4.2.2. Let μ be a point process on $\mathbb{R}^d$ that admits m -correlation function. We call $\mathfrak{d}_\mu^{[m]} = (\mathfrak{d}_{\mu;i,n}^{[m]})_{i=1,\dots,d;n=1,\dots,m}$ the m -logarithmic derivative of μ if

$$\mathfrak{d}_\mu^{[m]} \in \{L_{loc}^1((\mathbb{R}^d)^m \times \text{Conf}(\mathbb{R}^d), \mathcal{C}_\mu^{[m]})\}^{dm}$$

and, for each $\varphi(x_1, \dots, x_m, \xi) \in C_0^\infty((\mathbb{R}^d)^m) \otimes \mathcal{D}_o$,

$$\int_{(\mathbb{R}^d)^m \times \text{Conf}(\mathbb{R}^d)} \nabla_{d,m} \varphi d\mathcal{C}_\mu^{[m]}(x_1, \dots, x_m, \xi) = - \int_{(\mathbb{R}^d)^m \times \text{Conf}(\mathbb{R}^d)} \varphi \mathfrak{d}_\mu^{[m]} d\mathcal{C}_\mu^{[m]}(x_1, \dots, x_m, \xi).$$

Here $\nabla_{d,m} \varphi = \left(\partial_{i,n} \varphi \right)_{i=1,\dots,d;n=1,\dots,m}$ and $\partial_{i,n} \varphi = \frac{\partial \varphi(x_1, \dots, x_m, \xi)}{\partial x_{i,n}}$.

Theorem 4.2.3. Let μ be a point process on $\mathbb{R}^d$ that admits an m -correlation function for each $m \in \mathbb{N}$. Assume that there exists an m -logarithmic derivative of μ for each $m \in \mathbb{N}$. Then μ is Gibbsian.

Assume that μ admits an m -correlation function for each $m \in \mathbb{N}$ and 1-logarithmic derivative. Then μ admits m -logarithmic derivative for each $m \in \mathbb{N}$ of the form

$$\mathfrak{d}_\mu^{[m]}(x_1, \dots, x_m, \xi) = \left(\mathfrak{d}_\mu^{[1]}(x_i, \sum_{j \neq i}^m \delta_{x_j} + \xi) \right)_{i=1}^m.$$

Hence we have:

Corollary 4.2.4. *Let μ be a point process on $\mathbb{R}^d$ that admits m -correlation function for each $m \in \mathbb{N}$. Assume that there exists a 1-logarithmic derivative of μ . Then μ is Gibbsian.*

Next we introduce a symmetric form $(\mathcal{E}, \mathcal{D})$ on $L^2(\text{Conf}(\mathbb{R}^d), \mu)$ as in [18]. Denote for $m \in \mathbb{N} \cup \{\infty\}$

$$\mathbf{D}_m[f, g](x) = \frac{1}{2} \sum_{n=1}^m \sum_{i=1}^d \partial_{i,n} f(x) \partial_{i,n} g(x).$$

Set $\text{Conf}(\mathbb{R}^d)_m = \{\xi \in \text{Conf}(\mathbb{R}^d); \xi(\mathbb{R}^d) = m\}$ for $m \in \mathbb{N} \cup \{\infty\}$. For $\phi, \psi \in \mathcal{D}_\circ$, we set $\mathbf{D}[\phi, \psi] : \text{Conf}(\mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathbf{D}[\phi, \psi](\xi) &= \mathbf{D}_m[\phi_m(\mathbf{l}_m(\xi)), \psi_m(\mathbf{l}_m(\xi))] \quad \text{if } \xi \in \text{Conf}(\mathbb{R}^d)_m, m \in \mathbb{N} \cup \{\infty\} \\ &= 0 \quad \text{if } \xi(\mathbb{R}^d) = 0. \end{aligned}$$

Here ϕ_m is defined in (4.2.3) and $\mathbf{l}_m : \text{Conf}(\mathbb{R}^d)_m \rightarrow (\mathbb{R}^d)^m$ is a map such that $\mathbf{l}_m(\xi) = (\mathbf{l}_m^1(\xi), \mathbf{l}_m^2(\xi), \dots, \mathbf{l}_m^m(\xi))$ and $\xi = \sum_{n=1}^m \delta_{\mathbf{l}_m^n(\xi)}$. Set $(\mathcal{E}, \mathcal{D}) = (\mathcal{E}^\mu, \mathcal{D}^\mu)$ by

$$\begin{aligned} \mathcal{E}(\phi, \psi) &= \int_{\text{Conf}(\mathbb{R}^d)} \mathbf{D}[\phi, \psi](\xi) \mu(d\xi), \\ \mathcal{D} &= \{\phi \in \mathcal{D}_\circ \cap L^2(\text{Conf}(\mathbb{R}^d), \mu); \mathcal{E}(\phi, \phi) < \infty\}. \end{aligned}$$

Theorem 4.2.5. *Let μ be a point process on $\mathbb{R}^d$ that admits an m -correlation function for each $m \in \mathbb{N}$. Assume that μ is Gibbsian and, for each $R, m \in \mathbb{N}$, the Radon-Nikodym density $d\tilde{\mu}_{R,m,\eta}/dx$ is continuous on $(B_R)^m$ for μ -a.s. η . Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\text{Conf}(\mathbb{R}^d), \mu)$.*

Especially in the case $d = 1$, above theorem works powerfully because the existence of a 1-logarithmic derivative implies the closability of the form.

Theorem 4.2.6. *Let μ be a point process on $\mathbb{R}$ that admits an m -correlation function for each $m \in \mathbb{N}$. Assume that there exists a 1-logarithmic derivative of μ . Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\text{Conf}(\mathbb{R}), \mu)$.*

4.3 Proof of Theorem 4.2.3

Lemma 4.3.1 (Lemma 3.2.10 in [13]). *Let θ be a finite Borel measure on $\mathbb{R}^d$. Assume that there exists a constant C such that for each $\phi \in C_0^\infty(\mathbb{R}^d)$*

$$\left| \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial x_i}(x) d\theta(x) \right| \leq C \sup_{x \in \mathbb{R}^d} |\phi(x)|, \quad i = 1, \dots, d.$$

Then θ is absolutely continuous with respect to the Lebesgue measure.

Define the measure $P_\xi^{[m]}(dx)$ and the probability measure $Q^{[m]}(d\xi)$ by disintegration

$$\mathcal{C}_\mu^{[m]}(A \times B) = \int_{A \times B} P_\xi^{[m]}(dx) Q^{[m]}(d\xi). \quad (4.3.1)$$

Then $P_\xi^{[m]}$ is a σ -finite measure for $Q^{[m]}(\xi)$ -a.s. ξ . Set for $Q^{[m]}(\xi)$ -a.s. ξ

$$P_{\xi,R}^{[m]}(dx) = 1_{(B_R)^m}(x) P_\xi^{[m]}(dx).$$

Then $P_{\xi,R}^{[m]}$ is a finite measure on $(\mathbb{R}^d)^m$ for each $R \in \mathbb{N}$.

Lemma 4.3.2. *Let μ be a point process on $\mathbb{R}^d$. Assume that $\xi^{[m]}(B_R) \in L^1(\text{Conf}(\mathbb{R}^d), \mu)$ for each $R \in \mathbb{N}$ and there exists an m -logarithmic derivative of μ . Then $P_{\xi,R}^{[m]}$ is absolutely continuous with respect to the Lebesgue measure on $(B_R)^m$ for μ -a.s. ξ and each $R \in \mathbb{N}$.*

Proof. By Definition 4.2.2 and (4.3.7), for each $f(x)g(\xi) \in C_0^\infty((\mathbb{R}^d)^m) \otimes \mathcal{D}_o$,

$$\int_{\text{Conf}(\mathbb{R}^d)} \left(\int_{(\mathbb{R}^d)^m} \partial_{i,n} f(x) + f(x) \mathfrak{d}_{\mu;i,n}^{[m]}(x, \xi) P_\xi^{[m]}(dx) \right) g(\xi) Q^{[m]}(d\xi) = 0.$$

Because $\mathcal{D}_o \subset L^2(\text{Conf}(\mathbb{R}^d), \mu)$ and $\mathcal{D}_o$ is dense in $C_b(\text{Conf}(\mathbb{R}^d))$, we have for $Q^{[m]}$ -a.e. ξ and each $f \in C_0^\infty((\mathbb{R}^d)^m)$,

$$\int_{(\mathbb{R}^d)^m} \partial_{i,n} f(x) + f(x) \mathfrak{d}_{\mu;i,n}^{[m]}(x, \xi) P_\xi^{[m]}(dx) = 0. \quad (4.3.2)$$

Hence, for any relatively compact set $A \subset (\mathbb{R}^d)^m$ and any $f \in C_0^\infty((\mathbb{R}^d)^m)$

$$\begin{aligned} \left| \int_A \partial_{i,n} f(x) P_\xi^{[m]}(dx) \right| &\leq \int_A |f(x)| |\mathfrak{d}_{\mu;i,n}^{[m]}(x, \xi)| P_\xi^{[m]}(dx) \\ &\leq \sup_{x \in (\mathbb{R}^d)^m} |f(x)| \int_A |\mathfrak{d}_{\mu;i,n}^{[m]}(x, \xi)| P_\xi^{[m]}(dx). \end{aligned} \quad (4.3.3)$$

By Definition 4.2.2 and Fubini's theorem, $\mathfrak{d}_{\mu;i,n}^{[m]}(\cdot, \xi) \in L_{loc}^1((\mathbb{R}^d)^m, P_\xi^{[m]})$. Let

$$C_{m,\xi,A} = \|\mathfrak{d}_{\mu;i,n}^{[m]}(\cdot, \xi)\|_{L^1(A, P_\xi)}.$$

Then (4.3.3) implies that for each $f \in C_0^\infty((\mathbb{R}^d)^m)$

$$\left| \int_A \partial_{i,n} f(x) P_\xi^{[m]}(dx) \right| \leq C_{m,\xi,A} \sup_{x \in (\mathbb{R}^d)^m} |f(x)|. \quad (4.3.4)$$

From (4.3.4) with $A = B_R$ combined with Lemma 4.3.1, $P_{\xi,R}^{[m]} = 1_{(B_R)^m}(x) P_\xi^{[m]}(dx)$ is absolutely continuous with respect to the Lebesgue measure. $\square$

For $m, R \in \mathbb{N}$ and $l \in \mathbb{N}_0$, define the probability measure $(B_R)^m \times \mathbf{Conf}(\mathbb{R}^d)_{R,l}$

$$\tilde{\mathcal{C}}_{\mu;R,l}^{[m]}(dx d\xi) = \tilde{C}_{\mu,m,R,l}^{-1} 1_{(B_R)^m}(x) 1_{\{\xi(B_R)=l\}}(\xi) \mathcal{C}_\mu^{[m]}(dx d\xi). \quad (4.3.5)$$

Here, $\tilde{C}_{\mu,m,R,l}$ is the normalizing constant. For $\eta \in \mathbf{Conf}(\mathbb{R}^d)$, define the conditional probability

$$\tilde{\mathcal{C}}_{\mu;R,l,\eta}^{[m]}(dx d\xi) = \tilde{\mathcal{C}}_{\mu;R,l}^{[m]}(dx d\xi \mid \xi_R^c = \eta_R^c). \quad (4.3.6)$$

By taking $l = 0$ in (4.3.6), we set the probability measure on $(B_R)^m$

$$\tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]}(dx) = \tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]}(dx d\xi).$$

Let $\mathbf{u}_m : (\mathbb{R}^d)^m \rightarrow \mathbf{Conf}(\mathbb{R}^d)$ be the delabeling map given by $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m \delta_{x_i}$. Define the symmetric measure $\check{\mu}_{R,m,\eta}$ on $(B_R)^m$ by the relation

$$\check{\mu}_{R,m,\eta} \circ \mathbf{u}_m^{-1} = \mu_{R,m,\eta}.$$

Then $\check{\mu}_{R,m,\eta}$ is a probability measure by construction.

Lemma 4.3.3. *Let μ be a point process on $\mathbb{R}^d$. For each $R, m \in \mathbb{N}$ and μ -a.s. η , we have*

$$\tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]} = \check{\mu}_{R,m,\eta}. \quad (4.3.7)$$

Proof. Recall that $\tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]}$ and $\check{\mu}_{R,m,\eta}$ are symmetric probability measures on $(B_R)^m$. Let $A \subset (B_R)^m$ be a Borel set. Assume that A is symmetric. By definition,

$$\tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]}(A) = \mu(\pi_R(\xi) \in \mathbf{u}_m(A) \mid \xi(B_R) = m, \xi_R^c = \eta_R^c). \quad (4.3.8)$$

Here, $\xi^{[m]}$ and $\mathbf{u}_m$ are defined in (4.2.2) and before (4.3.5), respectively.

On the other hand, by definition in (4.2.1)

$$\check{\mu}_{R,m,\eta}(A) = \mu(\pi_R(\xi) \in \mathbf{u}_m(A) \mid \xi(B_R) = m, \xi_R^c = \eta_R^c). \quad (4.3.9)$$

From (4.3.8) and (4.3.9), we obtain (4.3.7). $\square$

Proof of Theorem 4.2.3. Let $P_\eta^{[m]}$ be defined in (4.3.7). Define a probability measure on $(B_R)^m$ by

$$\tilde{P}_{\eta,R,0}^{[m]}(dx) = C_{\mu,m,\eta,R}^{-1} 1_{(B_R)^m}(x) P_{\eta_R^c}^{[m]}(dx).$$

Here, $\eta_R^c = \eta(\cdot \cap B_R^c)$. Then by definition,

$$\tilde{P}_{\eta,R,0}^{[m]}(dx) = \tilde{\mathcal{C}}_{\mu;R,0,\eta}^{[m]}(dx).$$

From this combined with Lemma 4.3.3, for μ -a.s. $\eta \in \mathbf{Conf}(\mathbb{R}^d)$

$$\tilde{P}_{\eta,R,0}^{[m]}(dx) = \check{\mu}_{R,m,\eta}(dx). \quad (4.3.10)$$

Then by Lemma 4.3.2 we obtain the claim. $\square$

4.4 Proof of Theorem 4.2.5 and Theorem 4.2.6

We modify Lemma 3.2. in [18]:

Lemma 4.4.1. *Let λ be a continuous function on $(B_R)^m \subset (\mathbb{R}^d)^m$. Denote $\lambda(dx) = \lambda(x)dx$. Let*

$$\begin{aligned}\mathcal{E}_\lambda(f, g) &= \int_{(B_R)^m} \mathbf{D}_m[f, g](x) \lambda(dx), \\ \mathcal{D}_\lambda &= \{f \in C_b^\infty((B_R)^m) \cap L^2((B_R)^m, \lambda) ; \mathcal{E}_\lambda(f, f) < \infty\}.\end{aligned}$$

Then $(\mathcal{E}_\lambda, \mathcal{D}_\lambda)$ is closable on $L^2((B_R)^m, \lambda)$.

Proof. Let $O_n = \{x \in (B_R)^m ; \frac{1}{n} < \lambda(x) < n\}$ and

$$\begin{aligned}\mathcal{E}_{\lambda,n}(f, g) &= \int_{O_n} \mathbf{D}_m[f, g](x) \lambda(dx), \\ \mathcal{E}_n(f, g) &= \int_{O_n} \mathbf{D}_m[f, g](x) dx.\end{aligned}$$

Then by definition, for each $f \in \mathcal{D}_\lambda$

$$\frac{1}{n} \mathcal{E}_n(f, f) \leq \mathcal{E}_{\lambda,n}(f, f) \leq n \mathcal{E}_n(f, f).$$

Because O_n is open, $(\mathcal{E}_n, \mathcal{D}_\lambda)$ is closable on $L^2((B_R)^m, \lambda)$. Hence $(\mathcal{E}_{\lambda,n}, \mathcal{D}_\lambda)$ is also closable on $L^2((B_R)^m, \lambda)$. Since $\{(\mathcal{E}_{\lambda,n}, \mathcal{D}_\lambda)\}$ is increasing sequence of closable forms, its limit $(\mathcal{E}_\lambda, \mathcal{D}_\lambda)$ is closable on $L^2((B_R)^m, \lambda)$. $\square$

Lemma 4.4.2 ([18]). *Let μ be a point process on $\mathbb{R}^d$. Let*

$$\mathcal{E}_{R,m,\eta}(f, g) = \int_{(B_R)^m} \mathbf{D}_m[f, g](x) \check{\mu}_{R,m,\eta}(dx).$$

Assume that, for each $m, R \in \mathbb{N}$, $(\mathcal{E}_{R,m,\eta}, C_b^\infty((B_R)^m))$ is closable on $L^2((B_R)^m, \check{\mu}_{R,m,\eta})$ for μ -a.s. η . Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\text{Conf}(\mathbb{R}^d), \mu)$.

Proof of Theorem 4.2.5. By assumption, $\check{\mu}_{R,m,\eta}$ is absolutely continuous with respect to the Lebesgue measure. Let $\mathbf{m}_{R,m,\eta}$ be the Radon-Nikodym density. By assumption, $\mathbf{m}_{R,m,\eta}$ is continuous on $(B_R)^m$. Hence by Lemma 4.4.1, $(\mathcal{E}, C_b^\infty((B_R)^m))$ is closable on $L^2((B_R)^m, \lambda)$. Then by Lemma 4.4.2, we obtain the claim. $\square$

Proof of Theorem 4.2.6. By Theorem 4.2.3, $\check{\mu}_{R,m,\eta}$ is absolutely continuous with respect to the Lebesgue measure. Let $\mathbf{m}_{R,m,\eta}$ be the Radon-Nikodym density. Then by (4.3.2) and (4.3.10), for each $f \in C_0^\infty(\mathbb{R}^m)$

$$\int_{(B_R)^m} \partial_{i,1} f(x) \mathbf{m}_{R,m,\eta}(x) dx = - \int_{(B_R)^m} f(x) \mathfrak{d}_{\mu;i,1}^{[m]}(x, \eta) \mathbf{m}_{R,m,\eta}(x) dx.$$

Because $\mathfrak{d}_{\mu;i,1}^{[m]}(\cdot, \eta) \in L_{loc}^1(\mathbb{R}^m, P_\eta^{[m]})$, for each $x \in (B_R)^m$ and $i \in \{1, \dots, m\}$

$$\mathfrak{d}_{\mu;i,n}^{[m]}(x, \eta) = \partial_{i,1} \log(\mathbf{m}_{R,m,\eta}(x)).$$

Hence $\mathbf{m}_{R,m,\eta}$ is continuous on $(B_R)^m$. Then by Theorem 4.2.5, we obtain the claim. $\square$

4.5 Application: determinantal point process on $\mathbb{R}$ related to Random matrices

Let μ be a point process on $\mathbb{R}^d$ that admits 1-correlation function ρ^1 . A reduced Palm measure μ_x of μ at $x \in \mathbb{R}^d$ is given by

$$\mu_x(\cdot) = \mu(\cdot - \delta_x \mid \xi(x) \geq 1).$$

Set $\rho^1(dx) = \rho^1(x)dx$ by the same symbol. The following assumption gives a sufficient condition for the existence of logarithmic derivative.

Assumption 1

1. $\rho^1(x) \in C^1(\mathbb{R}^d)$.
2. For $\rho^1(dx)$ -a.e. $x, y \in \mathbb{R}^d$, the reduced Palm measures μ_x and μ_y are equivalent.

Denote by $\mathcal{R}_{x,y}$ their Radon-Nikodym derivative as

$$d\mu_y(d\xi) = \mathcal{R}_{y,x}(\xi) d\mu_x(d\xi)$$

3. For $\rho^1(dx)$ -a.e. $x \in \mathbb{R}^d$, $\lim_{y \rightarrow x} \mathcal{R}_{y,x} = 1$ in $L^1(\text{Conf}(\mathbb{R}^d), \mu_x)$.

For a function $\phi \in C_0^\infty((\mathbb{R}^d)^m) \otimes \mathcal{D}_\circ$, we define the function $f_\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$f_\phi(\epsilon) = \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \mathcal{R}_{x+\epsilon, x}(\xi) \phi(x, \xi) d\mathcal{C}_\mu^{[1]}(x, \xi).$$

4. For any $\phi \in C_0^\infty((\mathbb{R}^d)^m) \otimes \mathcal{D}_\circ$, f_ϕ admits partial derivative in ϵ at $\epsilon = 0$. There exist functions $\partial_{i,1}\mathcal{R} : \mathbb{R}^d \times \mathbf{Conf}(\mathbb{R}^d) \rightarrow \mathbb{R}$, $i = 1, \dots, d$ such that for any $\phi \in C_0^\infty((\mathbb{R}^d)^m) \otimes \mathcal{D}_\circ$ and each $1 \leq i \leq d$, we have

$$\partial_{i,1}f_\phi(0) = \int_{\mathbb{R}^d \times \mathbf{Conf}(\mathbb{R}^d)} \partial_{i,1}\mathcal{R}(x, \xi) \phi(x, \xi) d\mathcal{C}_\mu^{[1]}(x, \xi).$$

Set

$$\nabla \mathcal{R} = (\partial_{1,1}\mathcal{R}, \dots, \partial_{d,1}\mathcal{R}).$$

Due to Proposition 2.2. in [2], we have:

Proposition 4.5.1 ([2]). *Let μ be a point process on $\mathbb{R}^d$ satisfying Assumption 1. Then $\rho^1(dx)$ -a.e. $x \in \mathbb{R}^d$, 1-logarithmic derivative $\mathfrak{d}_\mu^{[1]}$ exists and has the form*

$$\mathfrak{d}_\mu^{[1]}(x, \xi) = \nabla_{d,1} \log \rho^1(x) + \nabla \mathcal{R}(x, \xi).$$

When $d = 1$, Theorem 4.2.6 combined with Proposition 4.5.1 implies the following.

Theorem 4.5.2. *Let μ be a point process on $\mathbb{R}$ satisfying Assumption 1. Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\mathbf{Conf}(\mathbb{R}), \mu)$.*

Proof. By Proposition 4.5.1, μ has 1-logarithmic derivative. From this, combined with Theorem 4.2.6, we obtain the claim. $\square$

Especially, there is a wide class of determinantal point processes on $\mathbb{R}$ that satisfies Assumption 1. Here, we recall the definition of determinantal point process.

A point process μ on $\mathbb{R}$ is called a determinantal point process associated with a kernel $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ if its m -correlation function $\rho^m(x_1, \dots, x_m)$ is given by

$$\rho^m(x_1, \dots, x_m) = \det[K(x, y)]. \quad (4.5.1)$$

Let K be a positive definite Hermitian symmetric kernel on $\mathbb{R}$ that admits locally trace class operator with the spectrum between $[0, 1]$. Then by a theorem due to Machi[12], Shirai-Takahashi[25] and Soshnikov[24], there uniquely exists a determinantal point process on $\mathbb{R}$ associated with K . Note that by (4.5.1), $\rho^1(x) = K(x, x)$.

Let μ be a determinantal point process on $\mathbb{R}$ with kernel K .

Assumption 2

1. The operator associated with the kernel is an orthogonal projection onto a closed subset $L \subset L^2(\mathbb{R}, dx)$.
2. For $\rho^1(dx)$ -a.e. $x \in \mathbb{R}$, if $f \in L$ satisfies $f(a) = 0$ then $(x - a)^{-1}f \in L$.
3. $K(x, y) \in C^2(\mathbb{R}^2)$.
4. $\int_{\mathbb{R}} \frac{K(x, x)}{1+x^2} dx < \infty$.

The property in Assumption 2.2 is called the division property (cf. [3]). Remark that Assumption 2 is satisfied for the sine, Airy and Bessel kernel. More examples are found in [3, 4].

Take $y \in \mathbb{R}$, $R \gg 1$ and $\delta \ll 1$. Set the additive function $S_x^{R, \delta} : \mathbf{Conf}(\mathbb{R}) \rightarrow \mathbb{R}$ as for $\xi = \sum_n \delta_{x_n}$,

$$S_y^{R, \delta}(\xi) = \sum_{n: x_n \in B_R, |x_n - y| > \delta} \frac{2}{y - x_n}.$$

Set

$$\bar{S}_y^{R, \delta} = S_y^{R, \delta} - \mathbf{E}^{\mu_y}[S_y^{R, \delta}].$$

Then results in [1] implies that, under Assumption 2, for $\rho^1(dx)$ -a.e. $y \in \mathbb{R}$, there exists a function $\bar{S}_y : \mathbf{Conf}(\mathbb{R}) \rightarrow \mathbb{R}$ such that

$$\lim_{R \rightarrow \infty, \delta \rightarrow 0} \bar{S}_y^{R, \delta} = \bar{S}_y \text{ in } L^2(\mathbf{Conf}(\mathbb{R}), \mu_y).$$

Due to [1], Assumption 2 implies Assumption 1 with $\nabla \mathcal{R} = \bar{S}_y$.

Theorem 4.5.3 (Theorem 2.3 in [2]). *Let μ be a determinantal point process on $\mathbb{R}$ associated with a kernel $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ satisfying Assumption 2. Then for $\rho^1(dx)$ -a.e. $x \in \mathbb{R}$ the logarithmic derivative $\mathfrak{d}_\mu^{[1]}$ exists and has the form*

$$\mathfrak{d}_\mu^{[1]}(x, \xi) = \frac{d}{dx} \log \rho^1(x) + \bar{S}_x(\xi).$$

Due to Theorem 4.2.6 and Theorem 4.5.3, we obtain the follows.

Corollary 4.5.4. *Let μ be a determinantal point process on $\mathbb{R}$ that admits a kernel $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ satisfying Assumption 2. Then $(\mathcal{E}, \mathcal{D})$ is closable on $L^2(\mathbf{Conf}(\mathbb{R}), \mu)$.*

Bibliography

- [1] Bufetov, A. I. Quasi-symmetries of determinantal point processes. *Annals of Probability* 46, 956–1003 (2018).
- [2] Bufetov, A. I., Dymov, A. V. and Osada, H. The logarithmic derivative for point processes with equivalent Palm measures. *Journal of the Mathematical Society of Japan* 71, 451–469 (2019).
- [3] Bufetov, A. I. and Romanov, R. V. Division subspace and integrable kernels. *Bull. London Math. Soc.* 51 (2019) 267–277
- [4] Bufetov, A. I. and Shirai, T. Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces, *Proceedings of the Japan Academy Series A: Mathematical Sciences* 2017 vol: 93 (1) pp: 1-5
- [5] Bufetov, A. I., and Shirai, T. Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces, *Proc. Japan Acad. Ser. A Math. Sci.* Volume 93, Number 1 (2017), 1–5.
- [6] Georgii, H. O. and Yoo, H. J. Conditional intensity and gibbsianness of determinantal point processes. *Journal of Statistical Physics* 118, 55–84 (2005).
- [7] Hough, B., Krishnapur, M., Peres, Y., Virág, B. *Zeros of Gaussian analytic functions and determinantal point processes*, AMS, University Lecture Series **51**, 2009.
- [8] Liggett, T. M. : *Interacting Particle Systems*, Classics in Mathematics, Springer. (1985).
- [9] Lyons, R. Determinantal probability measures, *Publ. Math. Inst. Hautes Études Sci.*, **98** (2003), 167–212.
- [10] Lyons, R. Determinantal probability: basic properties and conjectures, arXiv in math 1406.2707v1 (2014).

- [11] Lyons, R., and Steif, J. E. : Stationary determinantal processes: phase multiplicity, bernoullicity, and domination, *Duke. Math. J.* **120** (2003), 515–575.
- [12] Macchi, O. (1975). The Coincidence Approach to Stochastic Point Processes. *Advances in Applied Probability*, 7(1), 83–122. doi:10.2307/1425855
- [13] Malliavin, P. *Integration and probability*. Springer, New York (1995)
- [14] Ornstein, D. S. : Bernoulli shifts with the same entropy are isomorphic, *Advances in Math.*, **4** (1970), 337–352.
- [15] Ornstein, D. S. : Two Bernoulli shifts with infinite entropy are isomorphic, *Advances in Math.* **5** (1970), 339–348.
- [16] Ornstein, D. S. : *Ergodic Theory, Randomness, and Dynamical Systems*, Yale Univ. Press, New Haven, Conn. (1974).
- [17] Ornstein, D. S., and Weiss, B. : Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math.* **48** (1987), 1–144.
- [18] Osada, H. Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions, *Comm. Math. Phys.* 176 (1996), no. 1, 117–131.
- [19] Osada, H. Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials, *Ann. Probab.* Volume 41, Number 1 (2013), 1–49.
- [20] Osada, H. and Osada, S. Discrete approximations of determinantal point processes on continuous spaces: tree representations and the tail triviality *J Stat Phys*, **170** (2018), 421–435.
- [21] Osada, H. and Tanemura, H. Infinite-dimensional stochastic differential equations and tail σ -fields, (2014), Preprint, arXiv:1412.8674.
- [22] Ruelle, D. *Statistical mechanics : rigorous results*. W.A. Benjamin, (1969).
- [23] Ruelle, D. Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18**, 127–159 (1970).
- [24] Soshnikov A. Determinantal random point fields, *Russian Math. Surveys*, **55** (2000), 923–975.
- [25] Shirai T., Takahashi Y. (2000) Fermion Process and Fredholm Determinant. In: Begehr H.G.W., Gilbert R.P., Kajiwara J. (eds) *Proceedings of the Second ISAAC Congress. International Society for Analysis, Applications and Computation*, vol 7. Springer, Boston, MA

- [26] Shirai T., and Takahashi Y. Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes, *J. Funct. Anal.* **205** (2003), 414–463.
- [27] Shirai T., and Takahashi Y. Random point fields associated with certain Fredholm determinants II: fermion shifts and their ergodic properties, *Ann. Prob.* **31** (2003), 1533–1564.
- [28] Steif, J. E. : Space-time Bernoullicity of the lower and upper stationary processes for attractive spin system, *Ann. Prob.* **19** (1991), No.2, 609–635.
- [29] Vere-Jones, D. A generalization of permanents and determinants, *Linear Algebra and its Applications* **111** (1988), 119-124.
- [30] Vere-Jones, D. Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions, *New Zealand J. Math.* **26**, (1997), 125–149.
- [31] Yoo, H. J. Gibbsianness of fermion random point fields. *Mathematische Zeitschrift* 252, 27–48 (2006).