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**Abstract**

The Tsuji formula for the Hall conductivity in metals is discussed in Haldane's framework.

The Tsuji formula [1] is widely known as a geometrical formula for the Hall conductivity in metals under weak magnetic field. Since it was derived under the assumption of the cubic symmetry, Haldane [2] tried to eliminate the assumption. Here we discuss the Tsuji formula using Haldane's framework. However, our conclusion is different from Haldane's. The details[1] are described in http://hdl.handle.net/2324/1957531.

In usual notation the weak-field DC Hall conductivity tensor $\sigma^{xy}$ per spin is given by [1, 2]

$$\sigma^{xy} = e^3 B \int \frac{\mathrm{d}S}{(2\pi)^3} \,(v^x,\, v^y) \begin{pmatrix} M^{-1}_{yy} & -M^{-1}_{yx} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^x \\ v^y \end{pmatrix} \frac{\tau^2}{|\vec{v}|}, \tag{1}$$

for the Fermi surface contribution in metals. Throughout this note we only consider the contribution from a single sheet of the Fermi surface. Here the magnetic field is chosen as $\vec{B} = (0, 0, B)$. The quasi-particle velocity $\vec{v} = (v^x, v^y, v^z)$ and the effective mass tensor $M_{\alpha\beta}$ are given by the derivative of the quasi-particle energy $\varepsilon$: $v^\alpha = \partial\varepsilon/\partial k^\alpha$ and $M^{-1}_{\alpha\beta} = \partial^2\varepsilon/\partial k^\alpha \partial k^\beta$. Since the contribution of the derivative of $\tau$ does not appear in the antisymmetric tensor $(\sigma^{xy} - \sigma^{yx})/2$, we have dropped it.

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[1]This note is a nutshell of our previous note, http://hdl.handle.net/2324/1957531.

Experimentally $\sigma^{xy}$ is obtained from the measurement where we measure the current in $x$-direction under the electric field in $y$-direction and the magnetic field in $z$-direction. If we measure the current in $y$-direction under the electric field in $x$-direction and the magnetic field in $z$-direction, we obtain $\sigma^{yx}$ described as

$$\sigma^{yx} = e^3 B \int \frac{\mathrm{d}S}{(2\pi)^3} \left(v^x,\, v^y\right) \begin{pmatrix} 0 & 0 \\ -M^{-1}_{xy} & M^{-1}_{xx} \end{pmatrix} \begin{pmatrix} v^x \\ v^y \end{pmatrix} \frac{\tau^2}{|\vec{v}|}. \tag{2}$$

Haldane [2] introduced the symmetric tensor $e^3 B\gamma_{zz} \equiv (\sigma^{xy} + \sigma^{yx})/2$. Eq. (1) and Eq. (2) lead to

$$\gamma_{zz} = \frac{1}{2} \int \frac{\mathrm{d}S}{(2\pi)^3} \left(v^x,\, v^y\right) \begin{pmatrix} M^{-1}_{yy} & -M^{-1}_{yx} \\ -M^{-1}_{xy} & M^{-1}_{xx} \end{pmatrix} \begin{pmatrix} v^x \\ v^y \end{pmatrix} \frac{\tau^2}{|\vec{v}|}. \tag{3}$$

Other symmetric tensors are introduced in the same manner as $e^3 B\gamma_{xx} \equiv (\sigma^{yz} + \sigma^{zy})/2$ and $e^3 B\gamma_{yy} \equiv (\sigma^{zx} + \sigma^{xz})/2$. As shown in the following the geometrical nature is captured by these symmetric tensors. It should be noted that our result, Eq. (3), is different form Haldane's [2]. The difference arises from the following fact. While Eq. (3) contains $(\partial v^x/\partial k^y)/|\vec{v}|$, Haldane erroneously uses $\partial(v^x/|\vec{v}|)/\partial k^y$ instead.

The target of our geometrical description is the mean curvature $H$ of the Fermi surface. It is given by

$$\begin{aligned} 2H &= \frac{1}{|\vec{v}|^3} \cdot \Big[\varepsilon_x\varepsilon_x(\varepsilon_{yy} + \varepsilon_{zz}) + \varepsilon_y\varepsilon_y(\varepsilon_{zz} + \varepsilon_{xx}) + \varepsilon_z\varepsilon_z(\varepsilon_{xx} + \varepsilon_{yy}) \\ &- \varepsilon_x(\varepsilon_y\varepsilon_{yx} + \varepsilon_z\varepsilon_{zx}) - \varepsilon_y(\varepsilon_x\varepsilon_{xy} + \varepsilon_z\varepsilon_{zy}) - \varepsilon_z(\varepsilon_x\varepsilon_{xz} + \varepsilon_y\varepsilon_{yz})\Big], \end{aligned}$$

for any shape of the Fermi surface. Here we have used the notations $\varepsilon_\alpha \equiv v^\alpha$ and $\varepsilon_{\alpha\beta} \equiv M^{-1}_{\alpha\beta}$.

The geometrical information in our master equation, Eq. (3), is represented by $h_{zz}$ as

$$\gamma_{zz} = \int \frac{\mathrm{d}S}{(2\pi)^3} h_{zz}\tau^2,$$

with

$$h_{zz} = \frac{1}{2|\vec{v}|} \left(\varepsilon_x\varepsilon_x\varepsilon_{yy} + \varepsilon_y\varepsilon_y\varepsilon_{xx} - \varepsilon_x\varepsilon_y\varepsilon_{yx} - \varepsilon_y\varepsilon_x\varepsilon_{xy}\right).$$

Using

$$h_{xx} = \frac{1}{2|\vec{v}|} \left(\varepsilon_y\varepsilon_y\varepsilon_{zz} + \varepsilon_z\varepsilon_z\varepsilon_{yy} - \varepsilon_y\varepsilon_z\varepsilon_{zy} - \varepsilon_z\varepsilon_y\varepsilon_{yz}\right),$$

and

$$h_{yy} = \frac{1}{2|\vec{v}|} \left(\varepsilon_z\varepsilon_z\varepsilon_{xx} + \varepsilon_x\varepsilon_x\varepsilon_{zz} - \varepsilon_z\varepsilon_x\varepsilon_{xz} - \varepsilon_x\varepsilon_z\varepsilon_{zx}\right),$$

additionally, we obtain

$$\gamma_{zz} + \gamma_{xx} + \gamma_{yy} = \int \frac{\mathrm{d}S}{(2\pi)^3} H l^2, \tag{4}$$

with $l^2 = |\vec{v}|^2 \tau^2$. Our result, Eq. (4), is applicable to any shape of the Fermi surface. In the case of cubic symmetry Eq. (4) is reduced to the Tsuji formula [1, 2]

$$\gamma_{zz} = \gamma_{xx} = \gamma_{yy} = \int \frac{\mathrm{d}S}{(2\pi)^3} \frac{H}{3} l^2.$$

Experimentally $\gamma_{cc}$ is obtained from the measurements of $\sigma^{ab}$ and $\sigma^{ba}$ where $(c, a, b) = (z, x, y), (x, y, z), (y, z, x)$. By summing six experimental results with different configurations we can use Eq. (4).

# References


[1] M. Tsuji, J. Phys. Soc. Jpn. **13**, 979 (1958).

[2] F. D. M. Haldane, arXiv:cond-mat/0504227v2.