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Quaternion Analysis of Power Supply for Tokamak Plasma Control

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Abstract

Power supply for plasma control has been developed from condenser bank with ignitron for pulse tokamak plasma to PWM inverter with IGBT for steady tokamak plasma. Quaternion, four-dimensional hypercomplex number is good at describing three-dimensional rotation. Utilizing the performance of the quaternion rotation, we analyze three-phase power electronic circuit for the tokamak plasma control of equilibrium and stability. By further introduction of biquaternion concept, we can deal with phasor of each phase of three-phase AC similarly. But, we have to multiply the rotation operator from the left-hand side and from the right-hand side by using the conjugation. We try to verify the merits beyond the multiplication inconvenience from both sides. Not only symmetrical three-phase AC (positive sequence) but also negative sequence and zero sequence can be dealt with. Concerning quaternion power of three-phase AC, we can obtain the similar result as the one in pq theory. Quaternion can divide three-dimensional vector. The capability is utilized to develop matrix converter strategy based not only on Venturini method but also on space vector method in more detail.

Key words : *Quaternion, Conjugation, Biquaternion, Quaternion power, Tokamak plasma control, Ohm's law, Generator equation, Three-phase power electronic circuit, Matrix converter*

1. Introduction

Power supply for tokamak plasma control has been developed from condenser bank with ignitron for pulse tokamak plasma to PWM (Pulse Width Modulated) inverter with IGBT (Insulated Gate Bipolar Transistor) for steady tokamak plasma such as TRIAM-1M (superconducting Tokamak of Research Institute for Applied Mechanics) and QUEST (Q-shu University Experiment of Steady State Spherical Tokamak).

Quaternion, four-dimensional hypercomplex number is good at describing three-dimensional rotation as seen in three-dimensional game graphics programming theory¹⁾. Utilizing the performance, we analyze three-phase power electronic circuit by three-dimensional ro-

tation instead of transforming to two-dimensional rotation in alpha-beta coordinates. By the quaternion analysis, we become able to examine the detailed characteristics, which could not be seen in the alpha-beta coordinates. First, the similarity is shown. Second, the details of output voltage are clarified.

Quaternion, four-dimensional hyper-complex number, is good at dealing with description of three-dimensional rotation. But, we have to multiply rotation operator $\exp(+\hat{n}\theta)$ from the left-hand side (θ is rotation angle), even if the rotating axis $\hat{n}$ is perpendicular to the vector. Generally speaking, we must multiply $\exp(+\hat{n}\theta/2)$ from the left-hand side, and multiply $\exp(-\hat{n}\theta/2)$ from the right-hand side. Namely, conjugation is necessary. Therefore we should verify the merits despite of the multiplication inconvenience of quaternion concept.

Utilizing the quaternion characteristics, we analyze the phase rotation of three-phase AC voltage of power electronic circuit. By alpha-beta coordinate transforma-

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tion of three-phase AC voltage, we could deal with three-phase AC voltage as vector rotation in two-dimensional complex plane similarly as phasor of single-phase AC. And strategy of switching improvement in power electronics was clarified²⁾³⁾. Active and reactive powers of three-phase AC became able to be dealt with similarly as those of single-phase AC. When we apply the quaternion to three-phase AC, we expect that the merits of rotation and power in alpha-beta coordinates would be inherited.

Even if quaternion is applied to three-phase AC, merits such as rotation and power in alpha-beta coordinates are verified to be included. It is clarified that the projection of quaternion locus in three-dimensional space in the (1,1,1) direction is the same as alpha-beta transformation locus in two-dimensional space. By introduction of biquaternion concept, we can deal with phasor of each phase of three-phase AC similarly as in symmetrical coordinate method. Therefore, not only symmetrical three-phase AC (positive sequence) but also negative sequence and zero sequence can be dealt with. Concerning quaternion power of three-phase AC, we can obtain the similar result as the one in pq theory⁴⁾.

The quaternion concept is applied to analysis of matrix converter as an example of power electronic circuit for plasma control. Concerning direct matrix converter, we clarify the (1,1,1)-directional superposition of three-fold higher harmonics, which is necessary for improvement of voltage tranformation ratio. Concerning indirect matrix converter, we clarify the (1,1,1)-directional superposition of triangle-wave like three-fold harmonics, which is necessary for improvement of modulation index. Namely, by the quaternion analysis we can watch the characteristics of matrix converter, which could not be seen in the alpha-beta coordinates.

Quaternion can not only rotate three-dimensional vector but also divide three-dimensional vector. The capability should be utilized not only to analyze but also to develop matrix converter strategy based on space vector method in more detail.

2. Power Supply for Tokamak Plasma Control

In Advanced Fusion Research Center in Research Institute for Applied Mechanics, as a power supply for tokamak plasma control of equilibrium and stability, condenser bank was adopted for pulse tokamak plasma. Next for steady state tokamak plasma we adopted phase-controlled converter with thyristor and PWM inverter with GTO (Gate Turn-Off thyristor) in TRIAM-1M. Now we are going to adopt PWM inverter with IGBT in QUEST as shown in Fig. 1.

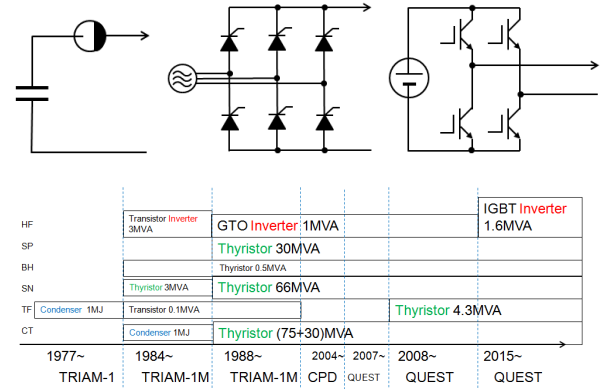


Fig. 1 Power supply for tokamak plasma control.

3. Quaternion Analysis of Three-Phase AC Circuit⁵⁾

3.1 Quaternion

Let's consider complex vector (phasor) representation and the exponential representation of single-phase AC current of effective value I .

$$i = \sqrt{2}I(\cos \omega t + j \sin \omega t) \quad (1)$$

Namely, they represent that effective current vector (including initial phase) rotates in counterclockwise on two-dimensional complex plane. The single-phase AC current is represented by the real part or imaginary part, and the phase angle is represented by the rotating angle from the positive real axis.

Next, let's consider space vector representation (polar coordinate representation of $\alpha\beta$ transformation) of symmetrical balanced (positive-sequence) three-phase AC phase (line) current.

$$\begin{aligned}
 i &= \sqrt{2/3}\sqrt{2}I\{\varepsilon^{j0\pi/3} \cos(\omega t - 0\pi/3) \\
 &\quad + \varepsilon^{j2\pi/3} \cos(\omega t - 2\pi/3) \\
 &\quad + \varepsilon^{j4\pi/3} \cos(\omega t - 4\pi/3)\} \\
 &= \sqrt{3}I\varepsilon^{j\omega t}
 \end{aligned} \quad (2)$$

Namely, it represents that the space current vector rotates in counterclockwise on the two-dimensional complex plane. The space current vector coincides with the synthesized rotating magnetic field, when three-phase AC currents energize electromagnets, which are located at the angle of $0, 2\pi/3$ and $4\pi/3$. Since we consider symmetrical balanced three-phase AC current, we may consider only the first phase rotates in counter clockwise. But notice that the coefficient just before the effective value is not $\sqrt{2}$ but $\sqrt{3}$.

Now, instead of transforming three-phase AC to two dimension, in order to represent three-phase AC in three dimension, let's introduce quaternion (four-dimensional

hypercomplex number), which is extended from a complex number⁶⁾.

$$q = a + iv_x + jv_y + kv_z = a + v \quad (3)$$

$$i^2 = j^2 = k^2 = -1 \quad (4)$$

$$\begin{cases} ij = -ji = k \\ jk = -kj = i \\ ki = -ik = j \end{cases} \quad (5)$$

Quaternion is divided into real part (scalar part) a and imaginary part (vector part) v , similarly with complex number. Namely, vector part has a property of vector, where imaginary numbers i, j, k behave as if they are unit base vectors, but they have also a property of hypercomplex number. The square of imaginary number i, j, k is equal to -1. The product of different imaginary numbers is the other imaginary number, and the sign depends on the order. And commutative law does not hold. To assign three-phase AC to the vector part, let's consider exponential representation of the quaternion.

$$\begin{aligned} q &= a + iv_x + jv_y + kv_z = a + v \\ &= a + \hat{n}\|v\| = \|q\|(\cos\theta + \hat{n}\sin\theta) = \|q\|\varepsilon^{\hat{n}\theta} \quad (6) \\ \begin{cases} \|v\|^2 = (v_x)^2 + (v_y)^2 + (v_z)^2 \\ \hat{n} = (+iv_x + jv_y + kv_z)/\|v\| \\ \|q\|^2 = a^2 + \|v\|^2 \end{cases} \quad (7) \end{aligned}$$

Quaternion can manipulate four dimensions, as it is interpreted as four-dimension number. But let the scalar part to be zero, and let's consider left-hand-side product of exponential hyper complex number ($\|q\| = 1, \hat{n} = k$) and a vector on xy plane.

$$\begin{aligned} &\varepsilon^{k\theta}(+iv_x + jv_y) \\ &= (\cos\theta + k\sin\theta)(+iv_x + jv_y) \\ &= +i(v_x\cos\theta - v_y\sin\theta) \\ &\quad + j(v_x\sin\theta + v_y\cos\theta) \quad (8) \end{aligned}$$

Namely, when exponential number is multiplied to the vector part from the left-hand side, the vector part rotates by θ in counterclockwise with an axis of the unit vector $\hat{n}$ as shown in Fig. 2(a). Here, the rotating axis must be perpendicular to the vector. When the vector has a component parallel to $\hat{n}$, the vector rotates on $(1, \hat{n})$ plane and the scalar part appears as shown in Fig. 2(b).

$$\begin{aligned} &\varepsilon^{k\theta}(+iv_x + jv_y + kv_z) \\ &= (\cos\theta + k\sin\theta)(+iv_x + jv_y + kv_z) \\ &= +i(v_x\cos\theta - v_y\sin\theta) \\ &\quad + j(v_x\sin\theta + v_y\cos\theta) \\ &\quad + kv_z\cos\theta - v_z\sin\theta \quad (9) \end{aligned}$$

Generally speaking, we must multiply $\varepsilon^{+\hat{n}\theta/2}$ from the left-hand side, and multiply $\varepsilon^{-\hat{n}\theta/2}$ from the right-hand

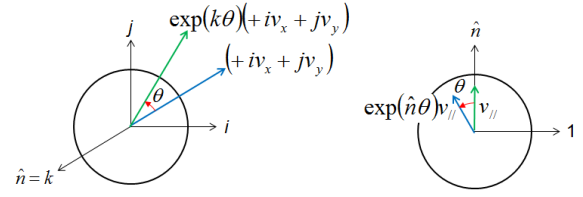


Fig. 2 Rotation of quaternion on (a) (i, j) and (b) $(1, k)$ plane.

side (conjugation is necessary) as follows.

$$\begin{aligned} &\varepsilon^{+k\theta/2}(+iv_x + jv_y + kv_z)\varepsilon^{-k\theta/2} \\ &= (\cos\theta/2 + k\sin\theta/2)(+iv_x + jv_y + kv_z) \\ &\quad (\cos\theta/2 - k\sin\theta/2) \\ &= \{+i(v_x\cos\theta/2 - v_y\sin\theta/2) \\ &\quad + j(v_x\sin\theta/2 + v_y\cos\theta/2) \\ &\quad + kv_z\cos\theta/2 - v_z\sin\theta/2\}(\cos\theta/2 - k\sin\theta/2) \\ &= +i(v_x\cos\theta - v_y\sin\theta) \\ &\quad + j(v_x\sin\theta + v_y\cos\theta) \\ &\quad + kv_z \quad (10) \end{aligned}$$

Let's assign three-phase AC phase (line-to-neutral) voltages to vector part of quaternion.

$$\begin{aligned} v &= \sqrt{2}V\{+i\cos(\omega t - 0\pi/3) \\ &\quad + j\cos(\omega t - 2\pi/3) \\ &\quad + k\cos(\omega t - 4\pi/3)\} \\ &= \varepsilon^{+\hat{n}\omega t/2}\sqrt{2}V_0\varepsilon^{-\hat{n}\omega t/2} \quad (11) \end{aligned}$$

$$\begin{cases} \hat{n} = (+i + j + k)/\sqrt{3} \\ V_0 = \{+i\cos(0 - 0\pi/3) \\ \quad + j\cos(0 - 2\pi/3) \\ \quad + k\cos(0 - 4\pi/3)\} \end{cases} \quad (12)$$

Namely, it represents that the initial three-phase (positive sequence) AC voltage vector V_0 rotates in counterclockwise with an axis of unit vector $\hat{n}$. In this case, the locus of the rotating vector is a circle on the plane, which is perpendicular to $\hat{n}$ and includes the origin. While we look at the xy plane from z axis in case of space vector, we look at the $(1,1,1)$ plane from the $(1,1,1)$ direction in case of quaternion (Fig. 3).

3.2 Ohm's Law

Let's consider Ohm's law of three-phase AC circuit, where the power supply is star-connected and resistor

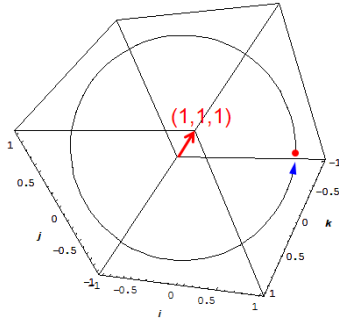


Fig. 3 Quaternion of symmetrical balanced 3-phase AC voltage.

load R is also star-connected.

$$\begin{aligned} & \sqrt{2} V \begin{bmatrix} \cos(\omega t - 0\pi/3) \\ \cos(\omega t - 2\pi/3) \\ \cos(\omega t - 4\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} R + R_N & R_N & R_N \\ R_N & R + R_N & R_N \\ R_N & R_N & R + R_N \end{bmatrix} \\ & \sqrt{2} I \begin{bmatrix} \cos(\omega t - 0\pi/3) \\ \cos(\omega t - 2\pi/3) \\ \cos(\omega t - 4\pi/3) \end{bmatrix} \end{aligned} \quad (13)$$

Since neutral resistance (ground resistance) R_N does not affect for the symmetrical three-phase (positive sequence) AC, the quaternion representation is as follows.

$$\begin{aligned} v &= \sqrt{2} V \{ +i \cos(\omega t - 0\pi/3) \\ & \quad + j \cos(\omega t - 2\pi/3) \\ & \quad + k \cos(\omega t - 4\pi/3) \} \\ &= \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} V V_0 \varepsilon^{-\hat{n}\omega t/2} \\ &= R \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} I I_0 \varepsilon^{-\hat{n}\omega t/2} \\ &= \varepsilon^{+\hat{n}\omega t/2} R \sqrt{2} I I_0 \varepsilon^{-\hat{n}\omega t/2} \end{aligned} \quad (14)$$

$$\begin{cases} \hat{n} = (+i + j + k)/\sqrt{3} \\ V_0 = \{ +i \cos(0 - 0\pi/3) \\ \quad + j \cos(0 - 2\pi/3) \\ \quad + k \cos(0 - 4\pi/3) \} \\ I_0 = \{ +i \cos(0 - 0\pi/3) \\ \quad + j \cos(0 - 2\pi/3) \\ \quad + k \cos(0 - 4\pi/3) \} \end{cases} \quad (15)$$

Next, let's consider Ohm's law of three-phase AC circuit, where the load is inductive and the mutual inductances exist.

$$\begin{aligned} & \sqrt{2} V \begin{bmatrix} \cos(\omega t + \phi - 0\pi/3) \\ \cos(\omega t + \phi - 2\pi/3) \\ \cos(\omega t + \phi - 4\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} L + L_N & L_N & L_N \\ L_N & L + L_N & L_N \\ L_N & L_N & L + L_N \end{bmatrix} p \\ & \sqrt{2} I \begin{bmatrix} \cos(\omega t - 0\pi/3) \\ \cos(\omega t - 2\pi/3) \\ \cos(\omega t - 4\pi/3) \end{bmatrix} \end{aligned} \quad (16)$$

Here, $p (= d/dt)$ is differential operator. Since the neutral inductance (grounding inductance) does not affect for symmetrical (positive sequence) AC, the quaternion representation is as follows.

$$\begin{aligned} v &= \sqrt{2} V \{ +i \cos(\omega t + \phi - 0\pi/3) \\ & \quad + j \cos(\omega t + \phi - 2\pi/3) \\ & \quad + k \cos(\omega t + \phi - 4\pi/3) \} \\ &= \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} V V_\phi \varepsilon^{-\hat{n}\omega t/2} \\ &= p(L - M) \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} I I_0 \varepsilon^{-\hat{n}\omega t/2} \\ &= \varepsilon^{+\hat{n}\omega t/2} \{ (+\hat{n}\omega/2)(L - M) \sqrt{2} I I_0 \\ & \quad + \sqrt{2} I I_0 (-\hat{n}\omega/2)(L - M) \} \varepsilon^{-\hat{n}\omega t/2} \end{aligned} \quad (17)$$

$$\begin{cases} \hat{n} = (+i + j + k)/\sqrt{3} \\ V_0 = \{ +i \cos(\phi - 0\pi/3) \\ \quad + j \cos(\phi - 2\pi/3) \\ \quad + k \cos(\phi - 4\pi/3) \} \\ I_0 = \{ +i \cos(0 - 0\pi/3) \\ \quad + j \cos(0 - 2\pi/3) \\ \quad + k \cos(0 - 4\pi/3) \} \end{cases} \quad (18)$$

Here, notice that the phase ϕ is $\pi/2$ in case of positive sequence.

Consequently, quaternion representation of general Ohm's law is as follows.

$$\begin{aligned} v &= \sqrt{2} V \{ +i \cos(\omega t + \phi - 0\pi/3) \\ & \quad + j \cos(\omega t + \phi - 2\pi/3) \\ & \quad + k \cos(\omega t + \phi - 4\pi/3) \} \\ &= \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} V V_\phi \varepsilon^{-\hat{n}\omega t/2} \\ &= \{ R + p(L - M) \} \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} I I_0 \varepsilon^{-\hat{n}\omega t/2} \\ &= \varepsilon^{+\hat{n}\omega t/2} \{ R \sqrt{2} I I_0 + (+\hat{n}\omega/2)(L - M) \sqrt{2} I I_0 \\ & \quad + \sqrt{2} I I_0 (-\hat{n}\omega/2)(L - M) \} \varepsilon^{-\hat{n}\omega t/2} \end{aligned} \quad (19)$$

$$\begin{cases} \hat{n} = (+i + j + k)/\sqrt{3} \\ V_\phi = \{ +i \cos(\phi - 0\pi/3) \\ \quad + j \cos(\phi - 2\pi/3) \\ \quad + k \cos(\phi - 4\pi/3) \} \\ I_0 = \{ +i \cos(0 - 0\pi/3) \\ \quad + j \cos(0 - 2\pi/3) \\ \quad + k \cos(0 - 4\pi/3) \} \end{cases} \quad (20)$$

Here, notice that the phase ϕ satisfies $\tan \phi = \omega(L - M)/R$ in case of positive sequence.

3.3 Biquaternion

Until this point, three-phase AC has been assigned to vector part of quaternion, similarly with space vector. Manipulation of three-phase AC has become easy, but is not convenient since each phase remains real time such as $\cos \omega t$ or $\sin \omega t$. In order to manipulate the three-phase AC similarly as complex vector (phasor) of single-phase AC and symmetrical coordinate method of three-phase AC, let's introduce biquaternion (eight-dimension

number), hyper complex number including quaternion.

$$h^2 = -1 \quad (21)$$

$$\begin{cases} ih = hi \\ jh = hj \\ kh = hk \end{cases} \quad (22)$$

Namely, we add a complex number h to the quaternion, which is exchangeable (commutative) with quaternion and independent from quaternion. And we assign complex vector representation and the exponential representation of each phase to the complex number h .

$$\begin{aligned} v &= \sqrt{2} V \{ +i\varepsilon^{h(\omega t + \phi - 0\pi/3)} \\ &\quad + j\varepsilon^{h(\omega t + \phi - 2\pi/3)} \\ &\quad + k\varepsilon^{h(\omega t + \phi - 4\pi/3)} \} \\ &= \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} V V_\phi \varepsilon^{-\hat{n}\omega t/2} \\ &= \{R + p(L - M)\} \varepsilon^{+\hat{n}\omega t/2} \sqrt{2} II_0 \varepsilon^{-\hat{n}\omega t/2} \\ &= \varepsilon^{+\hat{n}\omega t/2} \{R\sqrt{2} II_0 + (\hat{n}\omega/2)(L - M)\} \sqrt{2} II_0 \\ &\quad + \sqrt{2} II_0 (-\hat{n}\omega/2)(L - M) \} \varepsilon^{-\hat{n}\omega t/2} \end{aligned} \quad (23)$$

$$\begin{cases} \hat{n} = (+i + j + k)/\sqrt{3} \\ V_\phi = \{ +i\varepsilon^{h(\phi - 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi - 2\pi/3)} \\ \quad + k\varepsilon^{h(\phi - 4\pi/3)} \} \\ I_0 = \{ +i\varepsilon^{h(0 - 0\pi/3)} \\ \quad + j\varepsilon^{h(0 - 2\pi/3)} \\ \quad + k\varepsilon^{h(0 - 4\pi/3)} \} \end{cases} \quad (24)$$

Namely, let's combine the effective value and initial bi-quaternion into biquaternion, and we can obtain the same representation as the complex vector representation of single-phase AC. And we have only to use unit pure quaternion $\hat{n}$ instead of imaginary number j .

3.4 Equation for Generator

Positive-sequence AC voltage of effective value E_a is generated in three-phase AC generator. Equation for generator is as follows.

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ E_a \\ 0 \end{bmatrix} - \begin{bmatrix} Z_0 I_0 \\ Z_1 I_1 \\ Z_2 I_2 \end{bmatrix} \quad (25)$$

Here, V_0, V_1 and V_2 are zero-, positive- and negative-sequence voltage, respectively. Z_0, Z_1 and Z_2 are zero, positive- and negative-sequence impedance, respectively. I_0, I_1 and I_2 are zero-, positive- and negative-sequence current, respectively. We can obtain

the biquaternion representation as follows.

$$\begin{cases} \varepsilon^{+h\omega t} \sqrt{2} V_0 V_{0\phi} = 0 \\ \quad -\varepsilon^{+h\omega t} \{ (R + h\omega(L + 2M)) \sqrt{2} II_{00} \\ \varepsilon^{+\hat{n}\omega t} \sqrt{2} V_1 V_{1\phi} = \varepsilon^{+\hat{n}\omega t} \sqrt{2} E_a E_{10} \\ \quad -\varepsilon^{+\hat{n}\omega t} \{ (R + \hat{n}\omega(L - M)) \sqrt{2} II_{10} \\ \varepsilon^{-\hat{n}\omega t} \sqrt{2} V_2 V_{2\phi} = 0 \\ \quad -\varepsilon^{-\hat{n}\omega t} \{ (R - \hat{n}\omega(L - M)) \sqrt{2} II_{20} \end{cases} \quad (26)$$

$$\begin{cases} V_{00} = \{ +i\varepsilon^{h(\phi_{v0} - 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{v0} - 0\pi/3)} + k\varepsilon^{h(\phi_{v0} - 0\pi/3)} \} \\ V_{10} = \{ +i\varepsilon^{h(\phi_{v1} - 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{v1} - 2\pi/3)} + k\varepsilon^{h(\phi_{v1} - 4\pi/3)} \} \\ V_{20} = \{ +i\varepsilon^{h(\phi_{v2} + 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{v2} + 2\pi/3)} + k\varepsilon^{h(\phi_{v2} + 4\pi/3)} \} \end{cases} \quad (27)$$

$$\begin{aligned} E_{10} &= \{ +i\varepsilon^{h(0 - 0\pi/3)} \\ &\quad + j\varepsilon^{h(0 - 2\pi/3)} + k\varepsilon^{h(0 - 4\pi/3)} \} \end{aligned} \quad (28)$$

$$\begin{cases} I_{00} = \{ +i\varepsilon^{h(\phi_{i0} - 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{i0} - 0\pi/3)} + k\varepsilon^{h(\phi_{i0} - 0\pi/3)} \} \\ I_{10} = \{ +i\varepsilon^{h(\phi_{i1} - 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{i1} - 2\pi/3)} + k\varepsilon^{h(\phi_{i1} - 4\pi/3)} \} \\ I_{20} = \{ +i\varepsilon^{h(\phi_{i2} + 0\pi/3)} \\ \quad + j\varepsilon^{h(\phi_{i2} + 2\pi/3)} + k\varepsilon^{h(\phi_{i2} + 4\pi/3)} \} \end{cases} \quad (29)$$

Attention is necessary to the direction $\hat{n}$. Zero-sequence AC does not rotate but does oscillate along the $\hat{n}$. Attention is also necessary to the sign of of the inductive reactance in the negative sequence equation.

3.5 Quaternion Power

With quaternion, complex power is expressed as follows (Asterisk symbol in superfix means complex conjugate of quaternion)⁵⁾.

$$\begin{aligned} vi^* &= (+iv_a + jv_b + kv_c)(+ii_a + ji_b + ki_c)^* \\ &= (+iv_a + jv_b + kv_c) \\ &\quad (-ii_a - ji_b - ki_c) \\ &= (+v_a i_a + v_b i_b + v_c i_c) \\ &\quad -i(v_b i_c - v_c i_b) \\ &\quad -j(v_c i_a - v_a i_c) \\ &\quad -k(v_a i_b - v_b i_a) \end{aligned} \quad (30)$$

Concerning product of vector parts of quaternions, the scalar part means inner (scalar) product and the vector part means outer (vector) product. Namely, concerning quaternion power, the scalar part means the active power of three-phase (positive sequence) and the vector part means the reactive power. We notice that in space vector expression (positive sequence) the sum of

the vector part elements is obtained as follows.

$$\begin{aligned}
 v_p i_p^* &= (v_\alpha + jv_\beta)(i_\alpha + ji_\beta)^* \\
 &= \sqrt{2/3}(v_a \varepsilon^{+j0\pi/3} + v_b \varepsilon^{+j2\pi/3} + v_c \varepsilon^{+j4\pi/3}) \\
 &\quad \sqrt{2/3}(i_a \varepsilon^{-j0\pi/3} + i_b \varepsilon^{-j2\pi/3} + i_c \varepsilon^{-j4\pi/3}) \\
 &= (2/3)(+v_a i_a + v_b i_b + v_c i_c) \\
 &\quad + (2/3)(+v_a i_b + v_b i_c + v_c i_a) \varepsilon^{-j2\pi/3} \\
 &\quad + (2/3)(+v_b i_a + v_c i_b + v_a i_c) \varepsilon^{+j2\pi/3} \\
 &= (2/3)(+v_a i_a + v_b i_b + v_c i_c) \\
 &\quad + (1/3)(-v_a(i_b + i_c) - v_b(i_c + i_a) - v_c(i_a + i_b)) \\
 &\quad - j(1/\sqrt{3})(v_b i_c - v_c i_b) \\
 &\quad - j(1/\sqrt{3})(v_c i_a - v_a i_c) \\
 &\quad - j(1/\sqrt{3})(v_a i_b - v_b i_a)
 \end{aligned} \tag{31}$$

Here, if we assume the neutral current $i_a + i_b + i_c = 0$, or the neutral voltage $v_a + v_b + v_c = 0$, we obtain the following simple representation.

$$\begin{aligned}
 v_p i_p^* &= (+v_a i_a + v_b i_b + v_c i_c) \\
 &\quad - j(1/\sqrt{3})(v_b i_c - v_c i_b) \\
 &\quad - j(1/\sqrt{3})(v_c i_a - v_a i_c) \\
 &\quad - j(1/\sqrt{3})(v_a i_b - v_b i_a)
 \end{aligned} \tag{32}$$

Here, the real and imaginary part means instantaneous active and reactive power, respectively in pq theory⁴⁾. Notice the (1, 1, 1) component of the vector part of eq. (30) is the same as the imaginary part of eq. (32).

Similarly as a special case of Euler's four-square identity, a three-square identity holds good as a special case of no scalar part of quaternion. Namely, according to equation (30), square of apparent power (product of each quaternion norm) is sum of square of active power (norm of the first quaternion) and square of reactive power (norm the other quaternions)⁶⁾.

4. Quaternion Analysis of Matrix Converter

Three-phase to three-phase matrix converter (Fig. 4) is discussed, when the switching frequency is much higher than the modulation frequency. After introduction of quaternion concept, it is applied not only optimum Alesina-Venturini method but also space vector method.

4.1 Analysis of Direct Matrix Converter

Let's consider direct matrix converter, which obtains three-phase AC output voltage from three-phase AC input voltage without DC voltage. In case of Venturini method (AV method), three-phase output phase voltage becomes a circle, but the voltage transfer ratio is low to be 1/2. In case of improved Venturini method (optimum AV method) with improvement up to the

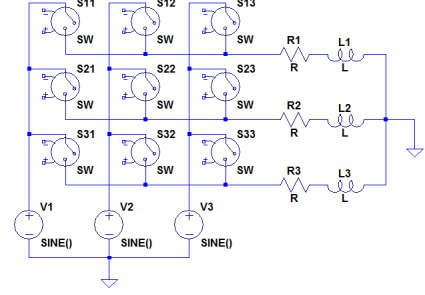


Fig. 4 Three-phase to three-phase matrix converter.

maximum $\sqrt{3}/2$, the three-phase output phase voltages are expressed as follows ($p = 0, 1, 2$ means the phase number)²⁾.

$$\begin{aligned}
 v_o &= q\sqrt{2} V_i \{ \cos(\omega_o t - p2\pi/3) \\
 &\quad - (1/6) \cos(3\omega_o t) + (1/2)\sqrt{3} \cos(\omega_i t) \}
 \end{aligned} \tag{33}$$

Namely, third higher harmonics of desired output phase voltage and input phase voltage are superimposed commonly to the three phases. Therefore, though the superimposed three-fold higher harmonics do not appear in the output line-to-line voltage, the neutral voltage oscillates in third higher harmonics. In case of $\omega_o = \pi, \omega_i = 2\pi$, three-phase output phase voltages are shown in Fig. 5 together with the case of Alesina-Venturini method⁷⁾.

$$\begin{aligned}
 v_o &= q\sqrt{2} V_i \{ +i \cos(\omega_o t - 0\pi/3) \\
 &\quad + j \cos(\omega_o t - 2\pi/3) + k \cos(\omega_o t - 4\pi/3) \\
 &\quad + (+i + j + k) \\
 &\quad \{ - (1/6) \cos(3\omega_o t) + (1/2)\sqrt{3} \cos(\omega_i t) \} \}
 \end{aligned} \tag{34}$$

Though the voltage transfer ratio is larger than that in case of Alesina-Venturini method, the quaternion does not draw a circle. Since the superimposed three-fold higher harmonics are common in three phases, quaternion oscillates in the direction $(+i + j + k) = \sqrt{3} \hat{n}$. The projection in the direction $(+i + j + k) = \sqrt{3} \hat{n}$ does not oscillate and the locus draws a circle. The space vector draws a circle in $\alpha\beta$ coordinate system, of course, and the quaternion of the output line-to-line voltage draws a circle in three-dimensional space.

In order to consider line-to-line voltages, let's consider product of $(+i + j + k)$ and voltage quaternion.

$$\begin{aligned}
 &(+i + j + k)(+iv_a + jv_b + kv_c) \\
 &= (+i + j + k) \\
 &\quad (+iv_a + jv_b + kv_c) \\
 &= -(+v_a + v_b + v_c) \\
 &\quad - \{ i(v_b - v_c) + j(v_c - v_a) + k(v_a - v_b) \}
 \end{aligned} \tag{35}$$

Namely, the scalar part means neutral voltage, and the vector part means the line-to-line voltages.

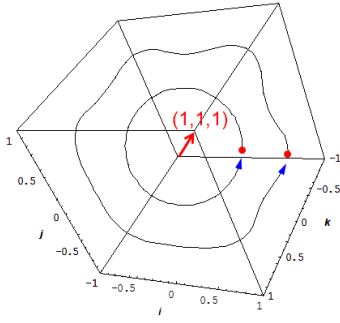


Fig. 5 Quaternion of output voltage. The inner circle is that of AV strategy and the outer curve is that of optimum AV strategy.

4.2 Analysis of Indirect Matrix Converter

Let's consider indirect matrix converter, which obtains three-phase AC output voltage from DC voltage, after obtaining DC voltage from three-phase AC input voltage. In the virtual inverter of the indirect matrix converter, we consider three-phase AC output phase voltage by space vector method. When switch changes from pnn configuration to ppn configuration, the relation between three-phase output phase voltages and intermediate DC voltage is as follows.

$$\begin{aligned} v_o &= \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \\ &= \begin{bmatrix} d_1 + d_2 + d_0/2 & 0 + d_0/2 \\ d_2 + d_0/2 & d_1 + d_0/2 \\ 0 + d_0/2 & d_2 + d_1 + d_0/2 \end{bmatrix} \begin{bmatrix} v_p \\ v_n \end{bmatrix} \end{aligned} \quad (36)$$

where d_1, d_2, d_0 are duty times of pnn, ppn, zero statuses, respectively. Though the number of switching is decreased, the three-phase output phase voltages for modulation index $m = \sqrt{3}/2$ are not sinusoidal waveform as shown in Fig. 6⁽⁸⁾. The average voltage (neutral voltage) behaves as triangle-wave like three-fold harmonics.

To consider the line-to-line voltages, we consider the product of a quaternion ($i + j + k$) to output phase voltage quaternion from left-hand side.

$$\begin{aligned} v_o &= -(d_2 - d_1) \\ &\quad -\{(k - j) \cos(\omega_o t) \\ &\quad + (i - k) \cos(\omega_o t - 2\pi/3) \\ &\quad + (j - i) \cos(\omega_o t - 4\pi/3)\} \end{aligned} \quad (37)$$

The neutral voltage (the scalar part) is triangle waveform approximately. The line-to-line voltages (the vector elements) are sinusoidal waves delayed by $2\pi/3$ from each other, and the line-to-line voltage quaternion draws a circle.

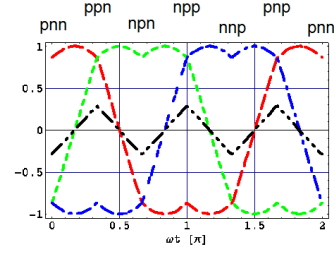


Fig. 6 Quaternion of output phase voltage of virtual inverter in space-vector approach. Black two-dotted dash curve indicates the neutral voltage.

4.3 Realization of Switching Matrix

When a switching matrix is expressed in dyadic form, we can deduce Alesina-Venturini method as a sum of constant matrix plus dyadic S composed of output voltage vector and input phase voltage vector.

$$\begin{aligned} V_o &= S V_i \\ V_o(V_i)^t &= S V_i(V_i)^t = S \end{aligned} \quad (38)$$

Usually in vector equation with coefficient matrix A , we can solve the unknown $x = A^{-1}y$ by obtaining the inverse matrix A^{-1} . But in complex number equation, we can calculate the transformation $C = v/u$ by dividing the number v by the number u . Similarly in switching equation for matrix converter, we can express the three-phase input and output voltages by using quaternion.

$$\varepsilon^{+\hat{n}\omega_o t} V_o = Q \varepsilon^{+\hat{n}\omega_i t} V_i \quad (39)$$

In this case, the switching quaternion Q is calculated as follows.

$$\begin{aligned} Q &= \varepsilon^{+\hat{n}\omega_o t} V_o(V_i)^{-1} \varepsilon^{-\hat{n}\omega_i t} \\ &= r \varepsilon^{+\hat{n}(\omega_o - \omega_i)t} \end{aligned} \quad (40)$$

Here, r is the voltage transfer ratio. The above quaternion equation is re-expressed by vector equation with switching matrix⁹⁾.

$$\varepsilon^{+\hat{n}\omega_o t} V_o = r \varepsilon^{+\hat{n}(\omega_o - \omega_i)t} \varepsilon^{+\hat{n}\omega_i t} V_i \quad (41)$$

$$\begin{aligned} V_o &= \begin{bmatrix} \cos(\omega_o t - 0\pi/3) \\ \cos(\omega_o t - 2\pi/3) \\ \cos(\omega_o t - 4\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} r \cos \omega_m t & -(r/\sqrt{3}) \sin \omega_m t & +(r/\sqrt{3}) \sin \omega_m t \\ +(r/\sqrt{3}) \sin \omega_m t & r \cos \omega_m t & -(r/\sqrt{3}) \sin \omega_m t \\ -(r/\sqrt{3}) \sin \omega_m t & +(r/\sqrt{3}) \sin \omega_m t & r \cos \omega_m t \end{bmatrix} \\ V_i &= \begin{bmatrix} \cos(\omega_i t - 0\pi/3) \\ \cos(\omega_i t - 2\pi/3) \\ \cos(\omega_i t - 4\pi/3) \end{bmatrix} \end{aligned} \quad (42)$$

$$\omega_m = \omega_o - \omega_i \quad (43)$$

For all components to be larger than 0 and smaller than 1, we have only to multiply 1/2 and add 1/2. But

that's too bad, the summation of three elements in each row is not constant and cannot be made unity by any means.

5. Summary

Even if quaternion is applied to three-phase AC, merits such as rotation and power in $\alpha\beta$ coordinates are verified to be included. It was clarified that the projection of quaternion locus in three-dimensional space in the (1,1,1) direction is $\alpha\beta$ transformation locus in two-dimensional space. By introduction of biquaternion concept, we can deal with phasor of each phase of three-phase AC similarly as in symmetrical coordinate method. Therefore, not only symmetrical three-phase AC (positive phase) but also negative phase and zero sequence can be dealt with. Concerning quaternion power of three-phase AC, we can obtain the similar result as the one in pq theory.

The quaternion concept was applied to analysis of matrix converter. Concerning direct matrix converter, we clarified the (1,1,1)-directional superposition of three-fold higher harmonics, which is necessary for improvement of voltage transformation ratio. Concerning indirect matrix converter, we clarified the (1,1,1)-directional superposition of triangle-wave like three-fold harmonics, which is necessary for improvement of modulation index. Namely, by the quaternion analysis we can watch the characteristics of matrix converter, which could not be seen in the alpha-beta coordinates.

Quaternion can not only rotate three-dimensional vector but also divide three-dimensional vector. When output voltage quaternion is divided by input one, switching quaternion is obtained. Though we could obtain switching matrix, whose element is larger than 0 and smaller than 1, we could not obtain, the one, sum of whose row elements is unity. The quaternion characteristics should be utilized to analyze matrix converter based on space vector method in more detail. Namely a new switching strategy has not been obtained from this quaternion analysis.

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References

- 1) Lengyel, E.: Mathematics for 3D Game Programming & Computer Graphics 2nd ed., *Charles River Media, Inc.*, (2004).
- 2) Wheeler, P. W., Rodriguez, J., Clare, J., Empringham, L., Weinstein, A.: Matrix Converters: A Technology Review, *IEEE Trans. on Industry Electronics*, vol. 49, no. 2, (2002) 274-288.
- 3) Casadei, D., Serra, G., Tani, A., Zarri, L.: Matrix Converter Modulation Strategies: A New General Approach Based on Space-Vector Representation of the Switch State, *IEEE Transactions on Industrial Electronics*, vol. 49, no. 2, (2002) 370-381.
- 4) Akagi, H., Kanazawa, Y., Fugita, K., Nanba, A.: Generalized Theory of the Instantaneous Reactive Power and its Application, *Trans. IEE of Japan*, vol. 103-B, no. 7, (1983) 483-490.
- 5) Nakamura, K., Fujita, H., Liu, X.L., Xue, E., Mitarai, O., Hasegawa, M., Tokunaga, K., Araki, K., Zushi, H., Hanada, K., Fujisawa, A., Idei, H., Nagashima, Y., Kawasaki, S., Nakashima, H., Higashijima, A.: Quaternion Analysis of Matrix Converter and Three-Phase Power Electronic Circuit, *International Conference on Electrical Engineering, ICEE 2014*, (2014) .
- 6) Conway, J. H., Smith, D.: On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry, *A. K. Peters, Ltd.*, (2003).
- 7) Nakamura, K., Fujita, H., Liu, X.L., Xue, E., Mitarai, O., Hasegawa, M., Tokunaga, K., Zushi, H., Hanada, K., Fujisawa, A., Idei, H., Nagashima, Y., Kawasaki, S., Nakashima, H., Higashijima, A., Araki, K.: Quaternion Concept in Matrix Converter Design for Feedback Stabilization by Magnetic Field Coil, *International Conference on Electrical Engineering, ICEE 2013*, (2013) FP0393.
- 8) Nakamura, K., Liu, X.L., Yoshisue, T., Mitarai, O., Tokunaga, K., Hasegawa, M., Zushi, H., Hanada, K., Fujisawa, A., Sakamoto, M., Idei, H., Kawasaki, S., Nakashima, H., Higashijima, A., Araki, K.: Venturini Method and Space Vector Modulation Method in Matrix Converter, *International Conference on Electrical Engineering, ICEE 2011*, (2011) A367.
- 9) Nakamura, K., Jamil, I., Liu, X.L., Mitarai, O., Hasegawa, M., Tokunaga, K., Araki, K., Zushi, H., Hanada, K., Fujisawa, A., Idei, H., Nagashima, Y., Kawasaki, S., Nakashima, H., Higashijima, A.: Quaternion Analysis of Three-Phase Power Electronic Circuit by Using Conjugation, *International Conference on Electrical Engineering, ICEE 2015*, (2015) ICEE15A-476.